mathematics

Department of physics level 2

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Chapter One

Sequence:-

A sequence is a function whose domain is the set of positive integer 1,2,3,… . We denote the sequence by $< f(n) >$) . Where $f(n)$ is the function at valued n.

The sequence f is the set $\{(n, f(n)); n \in N\}$. That is ,the set of all pairs $(n, f(n))$, with n appositive integer. We also write a_n to denote the sequence whose ordinate at $y = a_n$.

Examples:-

- $< \frac{1}{2n} > = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots \dots$

The set $\{ (n, \frac{1}{2n}) , n = 1, 2, 3, \dots .. \}$ is a sequence whose value at n is $\frac{1}{2n}$.

- $< \frac{1}{n+1} > = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \dots$

The set $\{ (n, \frac{1}{n+1}) , n = 1, 2, 3, \dots .. \}$ is a sequence whose value at n is $\frac{1}{n+1}$.

- $< 1 > = 1, 1, 1, \dots \dots ..$

The set $\{ (n, 1) , n = 1, 2, 3, \dots .. \}$ is a sequence whose value at n is 1.

Finite sequence :-

The sequence $< f(n) >$ is said to be finite if there exist two numbers a and b such that $a \leq f(n) \leq b \; or \; a < f(n) \leq b \; or \; a \leq f(n) < b \; or \; a < f(n) < b$.

Note:- every finite sequence is bounded.

Infinite sequence:-

The sequence $< f(n) >$ is said to be infinite if both a and b or one of them equal to $\pm\infty$.

Note:- every infinite sequence is unbounded.

Convergent Sequence:-

A sequence is said to be convergent if it approaches some limit .

Formally, a sequence S_n converges to the limit L . A divergent sequence doesn't have a limit. $\lim_{n\to\infty} S_n = L$

Discuss the following sequences:-

- $< \frac{\mathbf{1}}{\mathbf{2n}} > = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots \dots\} . n \in N$

The set $\{ (1,\frac{1}{2}), (2,\frac{1}{4}), (3,\frac{1}{6}), \dots \}$ is a finite and bounded sequence when

$n \to \infty,$

$$\lim_{n\to\infty} \frac{1}{2n} = 0$$

$< \frac{1}{2n} > \to 0$ the sequence converge to zero.

$$0 < \frac{1}{2n} \leq \frac{1}{2}$$

- $< \frac{\mathbf{1}}{\mathbf{n+1}} > = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \dots\}. n \in N$

The set $\{ (1,\frac{1}{2}), (2,\frac{1}{3}), (3,\frac{1}{4}), \dots \}$ is a finite and bounded sequence when

$n \to \infty,$

$$\lim_{n\to\infty} \frac{1}{n+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{n}{n}+\frac{1}{n}} = \frac{0}{1+0} = 0$$

$< \frac{1}{n+1} > \to 0$ the sequence is converge to zero.

$$0 < \frac{1}{n+1} \le \frac{1}{2}$$

- $< \mathbf{1} > = \{1, 1, 1, \ldots\ldots\ldots\ldots\}$.

The set $\{ (1,1), (2,1), (3,1), \ldots\ldots \}$ is a finite and bounded sequence when $n \to \infty$,

$$\lim_{n\to\infty} 1 = 1$$

$< 1 > \to 1$ the sequence is converge to 1.

- $< \frac{2n+1}{2n} > = \{\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots\ldots\ldots\}. n \in N$

The set $\{ (1,\frac{3}{2}), (2,\frac{5}{4}), (3,\frac{7}{6}), \ldots\ldots \}$ is a finite and bounded sequence since when $n \to \infty$, there are two Solutions for this sequence:-

First solution:-

$$\lim_{n\to\infty} \frac{2n+1}{2n} = \lim_{n\to\infty} \frac{\frac{2n}{n}+\frac{1}{n}}{\frac{2n}{n}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{2} = \frac{2+0}{2} = 1$$

$$\therefore < \frac{2n+1}{2n} > \to 1$$

Second solution:-

$$\lim_{n\to\infty} \frac{2n+1}{2n} = \lim_{n\to\infty} \frac{2n}{2n} + \frac{1}{2n} = \lim_{n\to\infty} 1 + \frac{1}{2n} = 1 + 0 = 1$$

$\therefore < \frac{2n+1}{2n} > \to 1$

The sequence is converge to 1.

$$1 < \frac{2n+1}{2n} \leq \frac{3}{2}$$

- $< \boldsymbol{n} > = \{1, 2, 3, \dots \dots\}. n \in N$

The set $\{ (1,1), (2,2), (3,3), \dots . . \}$ is a infinite and unbounded sequence when $n \to \infty$, $< n > \to \infty$. The sequence is diverged.

Note:- Those sequences are also infinite, unbounded and diverge $< \sqrt{\boldsymbol{n}} >, < \boldsymbol{5n^2} >, < \frac{\boldsymbol{n}^{3}}{\boldsymbol{6}} >, < \frac{\boldsymbol{n}^{6}}{\boldsymbol{n^3}} >$ And so on. Have a same solve like $< \boldsymbol{n} >$.

Example:- Find the Domains for the following sequences

- $< \frac{\boldsymbol{1}}{\boldsymbol{n-1}} >$

$n - 1 = 0$

$n = 1$

$n \in N/\{1\}$

Or $n = \{2, 3, 4, \dots . \}$.

- $< \sqrt{\boldsymbol{n - 4}} >$

$n - 4 \geq 0$

$n \geq 4$

$n \in N/\{1, 2, 3\}$

Or $n = \{4, t, 6, \ldots.\}$.

- $< \frac{\mathbf{1}}{\sqrt{\boldsymbol{n}+\mathbf{4}}} >$

$n \in N$ since $n + 4 > 0 \quad \rightarrow \quad n > -4$

- $< \frac{\mathbf{1}}{\sqrt{\boldsymbol{n}-\mathbf{3}}} >$

$n - 3 > 0$

$n > 3$

$n \in N/\{1, 2, 3\}$

Or $n = \{4, t, 6, \ldots.\}$.

- $< \frac{\mathbf{1}}{\sqrt{\mathbf{3}-\boldsymbol{n}}} >$

$3 - n > 0$

$3 > n$

$n = \{1, 2\}$.

$\therefore < \frac{1}{\sqrt{3-n}} >$ Is finite sequence having two limits.

- $< \sqrt{\mathbf{7} - \boldsymbol{n}} >$ How ?

Are the sequences converging or diverge?

1. $< \frac{\boldsymbol{n}-\mathbf{2}}{\boldsymbol{n}^{\mathbf{2}}+\mathbf{1}} >$

Solution:-

$$\lim_{n\to\infty} \frac{n-2}{n^2+1} = \lim_{n\to\infty} \frac{\frac{n}{n^2} - \frac{2}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{2}{n^2}}{1 + \frac{1}{n^2}} = \frac{0-0}{1+0} = 0$$

$$\therefore < \frac{n-2}{n^2+1} > \to 0$$

2. $< \frac{n^3+1}{n-2} >$

Sol.:- $\lim_{n\to\infty} \frac{n^3+1}{n-2} = \lim_{n\to\infty} \frac{\frac{n^3}{n^3}+\frac{1}{n^3}}{\frac{n}{n^3}-\frac{2}{n^3}} = \lim_{n\to\infty} \frac{1+\frac{1}{n^3}}{\frac{1}{n^2}-\frac{2}{n^3}} = \frac{1+0}{0-0} = \frac{1}{0} = \infty$

$\therefore < \frac{n^3+1}{n-2} >$ The sequence is diverge.

3. $< (-\mathbf{1})^{\boldsymbol{n}} >, n \in N$

Sol.:- $< (-1)^n > = \{-1, 1, -1, 1, \dots\}$

$$< (-1)^n > = \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

$$< (-1)^n > = \begin{cases} (1,-1), (3,-1), (t,-1), \dots & n \text{ odd} \\ (2,1), (4,1), (6,1), \dots & n \text{ even} \end{cases}$$

$$-1 \le (-1)^n \le 1$$

$\therefore < (-1)^n >$ Is bounded.

But $\lim_{n\to\infty} (-1)^n = \begin{cases} -1 & n \to \infty \text{ odd} \\ 1 & n \to \infty \text{ even} \end{cases}$

$\therefore < (-1)^n >$ Is diverge because it has two limits.

Discuss the following sequences? (Homework)

4. $< (-\mathbf{1})^{\boldsymbol{n+1}} >$

5. $< (-\mathbf{1})^{\boldsymbol{n-1}} >$

6. $< \mathbf{1} - (-\mathbf{1})^{\boldsymbol{n}} >$

7. $< \frac{1}{(-1)^n} >$

8. $< n - (-1)^n >$

9. $< \frac{n}{(-1)^n} >$

10. $< \frac{(-1)^n}{n} >$

11. $<< 5n + 4 >$

12. $< \frac{4}{3n} >$

13. $< \sqrt[5]{n} >$

14. $< \frac{n^2}{n^3+3} >$

Find the values of a_1, a_2, a_3 and a_4 for the following sequence $< a_n >$, were given a_n term ?

1. $a_n = \frac{1-n}{n^2}$

$$a_1 = \frac{1-1}{1^2} = \frac{0}{1} = 0$$

$$a_2 = \frac{1-2}{2^2} = \frac{-1}{4}$$

$$a_3 = \frac{1-3}{3^2} = \frac{-2}{9}$$

$$a_4 = \frac{1-4}{4^2} = \frac{-3}{16}$$

2. $a_n = \frac{(-1)^{n+1}}{2^{n-1}}$

$$a_1 = \frac{(-1)^{1+1}}{2^{1-1}} = \frac{(-1)^2}{2^0} = \frac{1}{1} = 1$$

$$a_2 = \frac{(-1)^{2+1}}{2^{2-1}} = \frac{(-1)^3}{2} = \frac{-1}{2}$$

$$a_2 = \frac{(-1)^{3+1}}{2^{3-1}} = \frac{(-1)^4}{2^2} = \frac{1}{4}$$

$$a_4 = \frac{(-1)^{4+1}}{2^{4-1}} = \frac{(-1)^5}{2^3} = \frac{-1}{8}$$

3. $\boldsymbol{a_n = \frac{(-1)^{n+1}}{2n-1}}$ **Homework?**

4. $\boldsymbol{a_n = (\frac{n+1}{2n})(1 - \frac{1}{n})}$ **Homework?**

Are the following sequence converge or diverge find the limited of each convergent sequence?

1. $\boldsymbol{a_n = 2 + (0.1)^n}$

$n = 1, 2, 3, ...$

$a_1 = 2.1$, $a_2 = 2.01$, $a_2 = 2.001, ...$

$n \to \infty$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(2 + (0.1)^n)$

$= \lim_{n\to\infty} (2 + \frac{1}{10^n})$

$= 2 + \lim_{n\to\infty} \frac{1}{10^n} = 2 + 0 = 2$

$\therefore a_n \to 2$

$2 \leq a_n \leq 2.1$

$< a_n >$ is bounded.

2. $\boldsymbol{a_n = 1 + (0.1)^n}$**? Homework**

Not :- All of these sequence are diverge:- $\boldsymbol{< a_n > = \frac{n}{10}}$, $\boldsymbol{< a_n > = (n-1)^2}$ **and** $\boldsymbol{< a_n > = (n + (-1)^n)}$.

when $n \to \infty$ the limit sequences $\lim_{n\to\infty} a_n = \infty$

3. $< a_n > = \frac{\cos n}{n}$

$-1 \le \cos n \le 1$

$$\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

But $< \frac{-1}{n} > \to 0$ and $< \frac{1}{n} > \to 0$

$\therefore < \frac{\cos n}{n} > \to 0$

By theorem if $a_n \le b_n \le c_n \ \forall n$ and $< a_n > \to L$ and $< c_n > \to L$ $then$ $< b_n > \to L.$

But this sequence is diverge show that $< a_n > = n + \cos n$ **?** **(Homework).**

Mathematical Convergent for Sequence:-

A sequence S_n is said to be converge Mathematically for a real number L if and only if for every positive real number ($\epsilon > 0$), there exist corresponding number ($K \in N$), such that :

$< S_n > \to L \leftrightarrow \ |S_n - L| < \epsilon \qquad \forall n > K.$

Example:- prove that

1. $< \frac{1}{n} > \to 0$

Proof:- $\left|\frac{1}{n} - 0\right| < \epsilon \qquad \forall n > K$

$|\frac{1}{n}| < \epsilon \qquad \forall n > K$

$\frac{1}{n} < \epsilon \qquad \forall n > K$

$n > \frac{1}{\epsilon} \qquad \forall n > K$

$\therefore K = \frac{1}{\epsilon}.$

2. $< \frac{1}{2^n} > \to 0$

Proof:- $\left|\frac{1}{2^n} - 0\right| < \epsilon \qquad \forall n > K$

$|\frac{1}{2^n}| < \epsilon \qquad \forall n > K$

$\frac{1}{2^n} < \epsilon \qquad \forall n > K$

$2^n > \frac{1}{\epsilon} \qquad \forall n > K$

$\ln 2^n > \ln \frac{1}{\epsilon}$

$n \ln 2 > \ln 1 - \ln \epsilon$

$n > \frac{0 - \ln \epsilon}{\ln 2}$

$\therefore K = \frac{-\ln \epsilon}{\ln 2}.$