CHAPTER ONE
MATRICES

Definition:

An array of **mn** numbers arranged in **m** rows and **n** columns is said to be **m×n** matrix.

j-th column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

i-th row

- **m×n** is the order (or size or dimension or degree) of the matrix
- The number which appears at the intersection of the i-th row and j-th column is usually referred as the (i,j)-th entry of the matrix A and denoted by a_{ij} and the matrix A denoted by $[a_{ij}]_{m\times n}$ or $A_{m\times n}$.
- The entries a_{ij} of a matrix A may be real or complex (or any field).
- If all entries of the matrix are real, then the matrix is called real matrix.
- If all entries of the matrix are complex, then the matrix is called complex matrix.
- If $A=[a_{ij}]_{m\times n}$ then $-A=[-a_{ij}]_{m\times n}$.

Examples:

(1) $A=\begin{bmatrix} 2 & -3 & 4 \\ 7 & 0 & -5 \end{bmatrix}$ is a real matrix, A has 2-rows and 3-columns.

$a_{11}=2$, $a_{12}=-3$, $a_{13}=4$, $a_{21}=7$, $a_{22}=0$, $a_{23}=-5$

(2) $B=\begin{bmatrix} 0 & i & \frac{1}{2} \\ 2+i & 4 & -3 \\ -1 & -i & 2+3i \end{bmatrix}$ is a complex matrix, B has 3-rows and 3-columns.

Types of Matrices

(1) Equal Matrices: The matrix $A = [a_{ij}]_{m\times n}$ equal to the matrix $B = [b_{ij}]_{m\times n}$ if they have the same size and the corresponding elements of A and B are equal $a_{ij} = b_{ij}$, $1 \le i \le m$ and $1 \le j \le n$.

Examples: (1) The matrices $\begin{bmatrix} 1 & -2 & 3 \\ 0 & \frac{3}{2} & 0.25 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1.5 & \frac{1}{4} \end{bmatrix}$ are equal matrices.

(2) The matrices $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 4 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 5 \\ 5 & 4 & 2 \end{bmatrix}$ are not equal matrices.

(2) Square matrix: A matrix having n rows and n columns and we say that it is of order n.

Examples: (1) $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}_{2\times 2}$ square matrix of order 2.

(2) $\begin{bmatrix} 5 & -6 & 9 \\ -3 & 2 & 1 \\ 0 & 8 & 0 \end{bmatrix}_{3\times 3}$ square matrix of order 3.

Note: The main diagonal in the square matrix contains the elements a_{ii} where $i = 1,2,\ldots,n$ (beginning from top left upper to bottom right).

(3) Zero matrix: the matrix all elements are zero denoted by $O_{m\times n}$.

Examples: (1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ square zero matrix of order 2.

(2) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{2\times 4}$ zero matrix of order 2×4.

(4) Identity Matrix: A square matrix all elements of the main diagonal are equal 1 and other elements are equal to 0 denoted by I_n.

Other Definition: A matrix $A=[a_{ij}]_{n\times n}$ is said to be identity matrix if

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}.$$

Examples: **(1)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ square identity matrix of order 2×2 (I_2).

(2) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ square identity matrix of order 4×4 (I_4).

(5) **Transpose of Matrix:** The matrix resulting from replacing the rows of the matrix by the column of it and denoted by A' or A^t or A^T.

Examples: **(1)** If $A = \begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & 1 & 5 & 7 \end{bmatrix}_{2\times 4}$ then $A^t = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}_{4\times 2}$.

(2) If $B = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 3 & 0 & -2 \\ 1 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix}_{5\times 3}$ then $B^t = \begin{bmatrix} 2 & -1 & 3 & 1 & 3 \\ 0 & 1 & 0 & 1 & 2 \\ -1 & 0 & -2 & -1 & 0 \end{bmatrix}_{3\times 5}$.

(6) **Symmetric Matrix:** A square matrix A is called symmetric matrix if it is equal to its transpose (i.e. $A = A^t$).

Examples: **(1)** $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $A^t = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ so A is symmetric matrix since $A = A^t$.

(2) $B = \begin{bmatrix} 6 & 3 & -2 \\ 3 & 0 & 5 \\ 2 & 5 & -4 \end{bmatrix}$ and $B^t = \begin{bmatrix} 6 & 3 & 2 \\ 3 & 0 & 5 \\ -2 & 5 & -4 \end{bmatrix}$ so B is not symmetric matrix since $B \neq B^t$.

(7) **Skew Symmetric Matrix:** A square matrix A is called skew symmetric matrix if it is equal to the negative of its transpose (i.e. $A = -A^t$).

Examples: (1) $A=\begin{bmatrix}0 & 2 & -1\\ -2 & 0 & 3\\ 1 & -3 & 0\end{bmatrix}$, $A^t=\begin{bmatrix}0 & -2 & 1\\ 2 & 0 & -3\\ -1 & 3 & 0\end{bmatrix}$ and $-A^t=\begin{bmatrix}0 & 2 & -1\\ -2 & 0 & 3\\ 1 & -3 & 0\end{bmatrix}$ so A is skew symmetric matrix since $A = -A^t$.

(2) $B=\begin{bmatrix}0 & -8\\ 8 & 1\end{bmatrix}$, $B^t=\begin{bmatrix}0 & 8\\ -8 & 1\end{bmatrix}$ and $-B^t=\begin{bmatrix}0 & -8\\ 8 & -1\end{bmatrix}$ so B is not skew symmetric matrix since $B \neq -B^t$.

Note: All the elements of the main diagonal in the skew symmetric matrix are zero.

(8) Conjugate Matrix: A matrix A is called conjugate matrix to the matrix B if the elements of the matrix A are the complex conjugated numbers with the elements of the matrix B and denoted by $\overline{A}$.

Examples: (1) The matrix $C=\begin{bmatrix}2i & 4\\ 1 & 3\end{bmatrix}$ is conjugate matrix to the matrix $D=\begin{bmatrix}-2i & 4\\ 1 & 3\end{bmatrix}$.

(2) The conjugate matrix to the matrix $A=\begin{bmatrix}2-i & 1+i & i\\ 2-3i & 4+i & 2i\\ 1-2i & -i & -2i\end{bmatrix}$ is the matrix $\overline{A}=\begin{bmatrix}2+i & 1-i & -i\\ 2+3i & 4-i & -2i\\ 1+2i & i & 2i\end{bmatrix}$

Note: The conjugate transpose of the matrix is equal to the transpose conjugate matrix.

i.e. $(\overline{A})^t=(\overline{A^t})=A^*$.

Example: $A=\begin{bmatrix}1+i & 3\\ -3 & -i\end{bmatrix} \Rightarrow A^t=\begin{bmatrix}1+i & -3\\ 3 & -i\end{bmatrix} \Rightarrow \left(\overline{A^t}\right)=\begin{bmatrix}1-i & -3\\ 3 & i\end{bmatrix}$

$$\overline{A}=\begin{bmatrix}1-i & 3\\ -3 & i\end{bmatrix} \Rightarrow \left(\overline{A}\right)^t=\begin{bmatrix}1-i & -3\\ 3 & i\end{bmatrix}=\left(\overline{A^t}\right)$$

(9) Hermitian Matrix: A square matrix A is called hermitian matrix if it is equal to its conjugate transpose (i.e. $A = A^*$).

Examples: **(1)** $A=\begin{bmatrix}4 & 2i\\ -2i & 1\end{bmatrix}$, $A^t=\begin{bmatrix}4 & -2i\\ 2i & 1\end{bmatrix}$ and $A^*=\overline{A^t}=\begin{bmatrix}4 & 2i\\ -2i & 1\end{bmatrix}$ so A is hermitian matrix since $A=A^*$.

(2) $B=\begin{bmatrix}2 & 1-2i & 2\\ 1-2i & 1 & -3i\\ 2 & 3i & 3\end{bmatrix}$, $B^t=\begin{bmatrix}2 & 1-2i & 2\\ 1-2i & 1 & 3i\\ 2 & -3i & 3\end{bmatrix}$ and

$B^*=\overline{B^t}=\begin{bmatrix}2 & 1+2i & 2\\ 1+2i & 1 & -3i\\ 2 & 3i & 3\end{bmatrix}$ so B is not hermitian matrix since $B\neq B^*$.

(10) Skew Hermitian Matrix: A square matrix A is called skew hermitian matrix if it is equal to its negative conjugate transpose (i.e. $A=-A^*$).

Note: All the elements of the main diagonal in the skew hermitian matrix are zero or pure imaginary numbers (complex number the real part of it must be equal to zero).

Examples :(1) $A=\begin{bmatrix}0 & -3i & 2\\ 3i & 0 & -i\\ 2 & i & 0\end{bmatrix}$, $\overline{A}=\begin{bmatrix}0 & 3i & 2\\ -3i & 0 & i\\ 2 & -i & 0\end{bmatrix}$ and $(\overline{A})^t=\begin{bmatrix}0 & -3i & 2\\ 3i & 0 & -i\\ 2 & i & 0\end{bmatrix}=A$

so A is not skew hermitian matrix since $A\neq -A^*$.

(2) $B=\begin{bmatrix}2i & 0\\ 0 & -3i\end{bmatrix}$, $\overline{B}=\begin{bmatrix}-2i & 0\\ 0 & 3i\end{bmatrix}$, $(\overline{B})^t=\begin{bmatrix}-2i & 0\\ 0 & 3i\end{bmatrix}$ and

$-(\overline{B})^t=\begin{bmatrix}2i & 0\\ 0 & -3i\end{bmatrix}=B$ so B is skew hermitian matrix since $B=-B^*$.

(11) Triangular Matrix: It is two kinds:

(*a*) Lower Triangular Matrix: A square matrix all its elements above the main diagonal are zeros, that is $a_{ij}=0$ for each i < j.

Examples: **(1)** $\begin{bmatrix}4 & 0 & 0\\ 2 & -1 & 0\\ 3 & 4 & 5\end{bmatrix}$ square lower triangular matrix of order 3×3.

(2) $B=\begin{bmatrix}5 & 0\\ 4 & -7\end{bmatrix}$ square lower triangular matrix of order 2×2.

(b) Upper Triangular Matrix: A square matrix all its elements below the main diagonal are zeros, that is $a_{ij} = 0$ for each i > j.

Examples: **(1)** $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ square upper triangular matrix of order 3×3.

(2) $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ square upper triangular matrix of order 2×2.

Note: the identity matrix is upper and lower triangular matrix.

(12) Diagonal Matrix: a square matrix A is called diagonal matrix if all elements are zero except the elements in the main diagonal (i.e. $a_{ij} = 0$ if $i \neq j$).

Other Definition: A square matrix which is upper and lower triangular matrix.

Examples: **(1)** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ square matrix of order 3×3, diagonal matrix, lower and upper triangular matrix, identity matrix.

(2) $B = \begin{bmatrix} 2 & 0 \\ 0 & -i \end{bmatrix}$ square matrix of order 2×2, diagonal matrix, lower and upper triangular matrix.

(13) Scalar Matrix: A diagonal matrix is called scalar matrix if all main diagonal elements are equal.

Other Definition: A diagonal matrix A = $[a_{ij}]_{n\times n}$ is called scalar matrix if $a_{11} = a_{22} = \ldots = a_{nn} = k$.

Examples: **(1)** $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ **(2)** $\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$

(14) Row Matrix: An 1×n matrix has one row $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$.

Examples: **(1)** $A = [\,3\ \ 2\ \ 1\ \ 4\,]_{1\times4}$ **(2)** $A = [\,7\ \ -5\ \ 2\ \ 3\ \ 1\,]_{1\times5}$

(15) Column Matrix: An $m\times1$ matrix has one column $A=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$.

Examples: **(1)** $B=\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}_{3\times1}$ **(2)** $B=\begin{bmatrix} -3 \\ 7 \\ 9 \\ 0 \end{bmatrix}_{4\times1}$

Exercises: Classify the following matrices according to their types

(1) $\begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & -2 \\ 4 & -2 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -4 \\ 0 & 4 & 5 \end{bmatrix}$

(4) $\begin{bmatrix} 0 & 1-2i & 5i \\ -1-2i & 0 & 3 \\ 5i & 3 & i \end{bmatrix}$

(5) $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix}$

(6) $\begin{bmatrix} i & i \\ i & i \end{bmatrix}$

(7) $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$

(8) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(9) $\begin{bmatrix} -3 & 1 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

(10) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(11) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(12) $\begin{bmatrix} 2 & 3 & -4 & 1 \end{bmatrix}$

(13) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 7 & 2 & 1 & 0 \end{bmatrix}$

(14) $\begin{bmatrix} 0 & -2 & -3 & -4 \\ 2 & 0 & -5 & -6 \\ 3 & 5 & 0 & -7 \\ 4 & 6 & 7 & 0 \end{bmatrix}$

(15) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

(16) $\begin{bmatrix} -1 \\ 6 \\ 8 \\ 0 \end{bmatrix}$

$$A + 2B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & \frac{2}{3} & -2 \\ 1 & 0 & -14 \end{bmatrix} = \begin{bmatrix} 5 & \frac{2}{3} & -1 \\ 3 & -1 & -11 \end{bmatrix}$$

$$A - 2B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & \frac{2}{3} & -2 \\ 1 & 0 & -14 \end{bmatrix} = \begin{bmatrix} -3 & -\frac{2}{3} & 3 \\ 1 & -1 & 17 \end{bmatrix}$$

Remarks:

(1) The subtraction of matrices is not commutative.

Example: If $A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -5 & 2 \end{bmatrix}$

$$A - B = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 7 & 2 \end{bmatrix}$$

$$B - A = \begin{bmatrix} 0 & 4 \\ -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ -7 & -2 \end{bmatrix}$$

So we get that $A - B \neq B - A$

(2) The subtraction of matrices is not associative.

Example: If $A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$, then

$(A - B) - C \neq A - (B - C)$ apply that. **(Home work)**

Theorem: For any two matrices of the same degree $B - A = -(A - B)$.

Proof: Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$

$-(A - B) = -([a_{ij}]_{m\times n} - [b_{ij}]_{m\times n})$

$= -[a_{ij} - b_{ij}]_{m\times n}$ (definition the subtraction of matrices)

$= -[c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij} - b_{ij}$

$= [-c_{ij}]_{m\times n}$

$= [-(a_{ij} - b_{ij})]_{m\times n}$ (replaced)

$= [-a_{ij} + b_{ij}]_{m\times n}$

$= [b_{ij} - a_{ij}]_{m\times n}$ (the addition of numbers is commutative)

$= [b_{ij}]_{m\times n} - [a_{ij}]_{m\times n}$ (definition the subtraction of matrices)

$= B - A$

Theorem: Let $M_{m\times n}(F)$ be the set of all m×n matrices over F, where F = ℝ or ℂ. Then for any scalars r, s and any A, B ∈ $M_{m\times n}(F)$

(1) r (A + B) = rA + rB

(2) (r + s) A = r A + sA

(3) r (sA) = (rs) A = s (rA)

(4) 1 A = A

(5) 0 A = O

(6) r A = A r

Proof (1): Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and r ∈ F

$r(A + B) = r([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n})$

$= r[a_{ij} + b_{ij}]_{m\times n}$ (definition the addition of matrices)

$= r[c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$

$= [r\,c_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [r(a_{ij} + b_{ij})]_{m\times n}$ (replaced)

$= [r\,a_{ij} + r\,b_{ij}]_{m\times n}$ (distribution of multiplication over the addition in numbers)

$= [r\,a_{ij}]_{m\times n} + [r\,b_{ij}]_{m\times n}$ (definition the addition of matrices)

$= r[a_{ij}]_{m\times n} + r[b_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

= rA + rB

Proof (2): Let $A = [a_{ij}]_{m\times n}$ and r, s ∈ F

$(r + s)A = (r + s)[a_{ij}]_{m\times n}$

$= [(r + s)\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [r\,a_{ij} + s\,a_{ij}]_{m\times n}$ (distribution of multiplication over the addition in numbers)

$= [r\,a_{ij}] + [s\,a_{ij}]_{m\times n}$ (definition the addition of matrices)

$= r[a_{ij}] + s[a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

= rA + sA

Proof (3): Let $A = [a_{ij}]_{m\times n}$ and r, s ∈ F

$r(sA) = r(s[a_{ij}])_{m\times n}$

$= r[s\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [r(s\,a_{ij})]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [(rs)\,a_{ij}]_{m\times n}$ (the multiplication of numbers is associative)

$= (rs)[a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

= (rs) A

$r\,(sA) = r\,(s[a_{ij}])_{m\times n}$

$= r\,[sa_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [r\,(sa_{ij})]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [(rs)a_{ij})]_{m\times n}$ (the multiplication of numbers is associative)

$= [(sr)\,a_{ij})]_{m\times n}$ (the multiplication of numbers is commutative)

$= [s\,(r\,a_{ij})]_{m\times n}$ (the multiplication of numbers is associative)

$= s\,[r\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= s\,(r\,[a_{ij}]_{m\times n})$ (definition the multiplication of matrix by scalar)

$= s\,(rA)$

Proof (4): Let $A = [a_{ij}]_{m\times n}$

$1\,A = 1\,[a_{ij}]_{m\times n} = [1\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [a_{ij}]_{m\times n}$

$= A$

Proof (5): Let $A = [a_{ij}]_{m\times n}$

$0\,A = 0\,[a_{ij}]_{m\times n} = [0\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= O_{m\times n}$

Proof (6): Let $A = [a_{ij}]_{m\times n}$, $r \in F$

$r\,A = r\,[a_{ij}]_{m\times n} = [r\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [a_{ij}\,r]_{m\times n}$ (the multiplication of numbers is commutative)

$= A\,r$

Multiplication of Matrices:

Two matrices are said to be compatible with multiplication if the number of the columns of the first matrix is equal to the number of the rows of the second matrix, i.e.

If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ then $AB = C = [a_{ij}]_{m\times n}\,[b_{ij}]_{n\times p} = \left[\sum_{k=1}^{n} a_{ik}\,b_{kj}\right]_{m\times p}$

Examples: **(1)** Let $A=\begin{bmatrix}1 & -2 & -1\\ 3 & 0 & -3\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0\\ -1 & 1\\ 0 & -2\end{bmatrix}$, find AB and BA?

Solution:

$$A\,B=\begin{bmatrix}1 & -2 & -1\\ 3 & 0 & -3\end{bmatrix}_{2\times 3}\begin{bmatrix}2 & 0\\ -1 & 1\\ 0 & -2\end{bmatrix}_{3\times 2}$$

$$=\begin{bmatrix}1(2)+(-2)(-1)+(-1)(0) & 1(0)+(-2)(1)+(-1)(-2)\\ 3(2)+0(-1)+(-3)(0) & 3(0)+0(1)+(-3)(-2)\end{bmatrix}=\begin{bmatrix}4 & 0\\ 6 & 6\end{bmatrix}_{2\times 2}$$

$$B\,A=\begin{bmatrix}2 & 0\\ -1 & 1\\ 0 & -2\end{bmatrix}_{3\times 2}\begin{bmatrix}1 & -2 & -1\\ 3 & 0 & -3\end{bmatrix}_{2\times 3}$$

$$=\begin{bmatrix}(2)(1)+0(3) & 2(-2)+0(0) & 2(-1)+0(-3)\\ (-1)(1)+1(3) & (-1)(-2)+1(0) & (-1)(-1)+1(-3)\\ 0(1)+(-2)(3) & 0(-2)+(-2)(0) & 0(-1)+(-2)(-3)\end{bmatrix}=\begin{bmatrix}2 & -4 & -2\\ 2 & 2 & -2\\ -6 & 0 & 6\end{bmatrix}_{3\times 3}$$

Note that $A\,B\neq B\,A$.

(2) $\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}_{2\times 2}\begin{bmatrix}1 & 0\\ -1 & 3\end{bmatrix}_{2\times 2}=\begin{bmatrix}1(1)+1(-1) & 1(0)+1(3)\\ 2(1)+1(-1) & 2(0)+1(3)\end{bmatrix}=\begin{bmatrix}0 & 3\\ 1 & 3\end{bmatrix}_{2\times 2}$

(3) Let $A=\begin{bmatrix}1 & 2\\ -1 & -2\\ 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 3\\ -1 & 5 & -2\end{bmatrix}$, find AB and BA if exist? **(Home work)**

(4) Let $A=\begin{bmatrix}1 & -2\\ 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 2\\ 1 & -2 & 1\end{bmatrix}$, find AB and BA if exist?

Solution:

$$AB=\begin{bmatrix}1 & -2\\ 2 & -1\end{bmatrix}_{2\times 2}\begin{bmatrix}2 & -1 & 2\\ 1 & -2 & 1\end{bmatrix}_{2\times 3}=\begin{bmatrix}1(2)+(-2)(1) & 1(-1)+(-2)(-2) & 1(2)+(-2)(1)\\ 2(2)+(-1)(1) & 2(-1)+(-1)(-2) & 2(2)+(-1)(1)\end{bmatrix}$$

$$=\begin{bmatrix}0 & 3 & 0\\ 3 & 0 & 3\end{bmatrix}_{2\times 3}$$

BA not exists since the number of B columns is not equal to the number of A rows.

Remarks: **(1)** Two matrices A and B are said to be commutative if AB = BA.

Example: Is the matrix $A=\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix}$ commutative with the matrix $B=\begin{bmatrix}7 & 8\\ 2 & 1\end{bmatrix}$?

Solution:

$$A\,B=\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix}_{2\times 2}\begin{bmatrix}7 & 8\\ 2 & 1\end{bmatrix}_{2\times 2}$$

$$=\begin{bmatrix}2(7)+4(2) & 2(8)+4(1)\\ 1(7)+(-1)(2) & 1(8)+(-1)(1)\end{bmatrix}=\begin{bmatrix}22 & 20\\ 5 & 7\end{bmatrix}_{2\times 2}$$

$$B\,A=\begin{bmatrix}7 & 8\\ 2 & 1\end{bmatrix}_{3\times 2}\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix}_{2\times 2}$$

$$=\begin{bmatrix}7(2)+8(1) & 7(4)+8(-1)\\ 2(2)+1(1) & 2(4)+1(-1)\end{bmatrix}=\begin{bmatrix}22 & 20\\ 5 & 7\end{bmatrix}_{2\times 2}$$

We get that AB = BA. So A and B are commutative matrices.

(2) The product of two matrices may be equal to zero matrix and each matrix is not a zero matrix.

Example: $AB=\begin{bmatrix}-1 & 1\\ -2 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}-1(2)+1(2) & -1(1)+1(1)\\ -2(2)+2(2) & -2(1)+2(1)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$

(3) The cancellation law not satisfies in matrices multiplication, i.e.

$AB = AC \not\Rightarrow B = C$

Example: Let $A=\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 5\\ 3 & 4\end{bmatrix}$

$$AB=\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix}=\begin{bmatrix}0(1)+1(3) & 0(1)+1(4)\\ 0(1)+2(3) & 0(1)+2(4)\end{bmatrix}=\begin{bmatrix}3 & 4\\ 6 & 8\end{bmatrix}$$

$$AC=\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\ 3 & 4\end{bmatrix}=\begin{bmatrix}0(2)+1(3) & 0(5)+1(4)\\ 0(2)+2(3) & 0(5)+2(4)\end{bmatrix}=\begin{bmatrix}3 & 4\\ 6 & 8\end{bmatrix}$$

AB = AC while B ≠ C.

(4) If both matrices A and B are square matrix of the same order with real entries, then it is not necessary that $(AB)^2 = A^2 B^2$.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -2 & -2 \end{bmatrix}$$

$$(AB)^2 = (AB)\,(AB) = \begin{bmatrix} -2 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 8 & 4 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$B^2 = BB = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A^2 B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -4 & 4 \end{bmatrix}$$

We get that $(AB)^2 \neq A^2 B^2$.

Note that this relation is true when the matrices are commutative, for example consider $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -3 \\ -2 & 2 \end{bmatrix}$ apply that $(AB)^2 = A^2 B^2$ **(Home work)**

Theorem: Let A be a matrix of degree m×n, then

(1) $A\,I_n = A$

(2) $I_m A = A$

(3) $AO = O$, $OA = O$

Proof (1): Let $A = [a_{ij}]_{m\times n}$, $I_n = [s_{ij}]_{n\times n}$ such that $s_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$$\begin{aligned} (i,j) \text{ element of } AI_n &= \sum_{k=1}^{n} a_{ik} s_{kj} \\ &= a_{i1}s_{1j} + a_{i2}s_{2j} + \ldots + a_{ij}s_{jj} + \ldots + a_{in}s_{nj} \\ &= a_{i1}(0) + a_{i2}(0) + \ldots + a_{ij}(1) + \ldots + a_{in}(0) \\ &= a_{ij}(1) \\ &= a_{ij} \\ &= (i,j) \text{ element of } A \end{aligned}$$

Degree of the matrix $A = m \times n$ = Degree of the matrix $A_{m\times n} I_{n\times n}$, so $A\,I_n = A$.

(15) Column Matrix: An $m\times1$ matrix has one column $A=\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$.

Examples: **(1)** $B=\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}_{3\times1}$ **(2)** $B=\begin{bmatrix} -3 \\ 7 \\ 9 \\ 0 \end{bmatrix}_{4\times1}$

Exercises: Classify the following matrices according to their types

(1) $\begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & -2 \\ 4 & -2 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -4 \\ 0 & 4 & 5 \end{bmatrix}$

(4) $\begin{bmatrix} 0 & 1-2i & 5i \\ -1-2i & 0 & 3 \\ 5i & 3 & i \end{bmatrix}$

(5) $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix}$

(6) $\begin{bmatrix} i & i \\ i & i \end{bmatrix}$

(7) $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$

(8) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(9) $\begin{bmatrix} -3 & 1 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

(10) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(11) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(12) $\begin{bmatrix} 2 & 3 & -4 & 1 \end{bmatrix}$

(13) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 7 & 2 & 1 & 0 \end{bmatrix}$

(14) $\begin{bmatrix} 0 & -2 & -3 & -4 \\ 2 & 0 & -5 & -6 \\ 3 & 5 & 0 & -7 \\ 4 & 6 & 7 & 0 \end{bmatrix}$

(15) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

(16) $\begin{bmatrix} -1 \\ 6 \\ 8 \\ 0 \end{bmatrix}$

Operations on Matrices

Addition of Matrices: Matrices are said to be compatible with addition if and only if they have the same degree.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices. The addition of A and B is denoted by A + B is also an m × n matrix, and

$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

$\quad = [a_{ij} + b_{ij}]_{m\times n}$

$\quad = [c_{ij}]_{m\times n}$ where $a_{ij} + b_{ij} = c_{ij}$ for all possible values of i, j

Examples:

(1) $$\begin{bmatrix} 1 & 5 \\ 4 & 6 \end{bmatrix}_{2\times 2} + \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1+2 & 5+0 \\ 4+(-1) & 6+0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 3 & 6 \end{bmatrix}_{2\times 2}$$

(2) $$A = \begin{bmatrix} 1 & 5 \\ 4 & 6 \end{bmatrix}_{2\times 2}, \; B = \begin{bmatrix} 1 & 5 \\ 4 & 6 \\ 2 & -1 \end{bmatrix}_{3\times 2}$$

A + B not define since A and B have different size. Thus A and B cannot be added.

(3) $$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & -2 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 & 3 \\ 2 & -3 & 5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & -2 \end{bmatrix}_{2\times 3} + \begin{bmatrix} -1 & 1 & 3 \\ 2 & -3 & 5 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 2+(-1) & 0+1 & (-1)+3 \\ 3+2 & 5+(-3) & (-2)+5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

$$B + A = \begin{bmatrix} -1 & 1 & 3 \\ 2 & -3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)+2 & 1+0 & 3+(-1) \\ 2+3 & (-3)+5 & 5+(-2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

We note that A + B = B + A. So Addition of matrices is commutative.

Properties of the Addition of Matrices

Theorem: Let $M_{m\times n}(F)$ be the set of all m×n matrices over F, where $F = \mathbb{R}$ or $\mathbb{C}$. Then:

(1) $A + B = B + A$ (The addition of matrices is commutative).

(2) $(A + B) + C = A + (B + C)$ (The addition of matrices is associative).

(3) $A + O = O + A = A$.

Proof (1): Let $A, B \in M_{m\times n}(F)$ such that $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, $a_{ij}, b_{ij} \in F$

$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [b_{ij} + a_{ij}]_{m\times n}$ (the addition of numbers is commutative)

$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$ (definition the addition of matrices)

$= B + A$

Proof (2): Let $A, B, C\in M_{m\times n}(F)$ such that $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and $C = [c_{ij}]_{m\times n}$, $a_{ij}, b_{ij}, c_{ij} \in F$

$(A + B) + C = ([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}) + [c_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [d_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$ where $d_{ij} = a_{ij} + b_{ij}$

$= [d_{ij} + c_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n}$ $d_{ij} = a_{ij} + b_{ij}$

$= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n}$ (the addition of numbers is associative)

$= [a_{ij}]_{m\times n} + [b_{ij} + c_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [a_{ij}]_{m\times n} + ([b_{ij}]_{m\times n} + [c_{ij}]_{m\times n})$ (definition the addition of matrices)

$= A + (B+ C)$

Proof (3): Let $A, O\in M_{m\times n}(F)$ such that $A=[a_{ij}]_{m\times n}$, $O= [b_{ij}]_{m\times n}$, $a_{ij}, \in F$, $b_{ij} = 0\ \forall$ i, j

$A + O = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [a_{ij} + 0]_{m\times n}$ $b_{ij} = 0\ \forall$ i, j

$= [a_{ij}]_{m\times n}$

$= A$

$O + A = [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$

$= [b_{ij} + a_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [0 + a_{ij}]_{m\times n}$ $b_{ij} = 0\ \forall$ i, j

$= [a_{ij}]_{m\times n}$

$= A$

Multiplication of Matrix by Scalar:

If A = $[a_{ij}]_{m\times n}$ matrix, c is a scalar then the scalar multiple c A is the m×n matrix obtained from A by multiplying each entry of A (which is a scalar too) by c. Thus cA = $[c\, a_{ij}]_{m\times n}$.

Examples:

(1) $2\begin{bmatrix} -1 & 0 & i \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2i \\ 6 & 4 & 0 \end{bmatrix}$

(2) $\frac{1}{4}\begin{bmatrix} 4 & -8 & 0 \\ 2 & 12 & 4 \\ 0 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 3 & 1 \\ 0 & \frac{3}{4} & 2 \end{bmatrix}$

(3) $i\begin{bmatrix} -1 & 0 & i \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 & -1 \\ 3i & 2i & 0 \end{bmatrix}$

Remark: If A = $[a_{ij}]_{m\times n}$, then (– 1) A = $[(-1)\, a_{ij}]_{m\times n} = [-\, a_{ij}]_{m\times n}$.
The matrix (– 1) A is denoted by – A and it is called the negative of A.
Also, A + (– A) = $O_{m\times n}$, i.e. – A is the additive inverse of A.

Proposition: Given A ∈ $M_{m\times n}(F)$, there exists B ∈ $M_{m\times n}(F)$ such that
A + B = $O_{m\times n}$ = B + A
In fact A determines B uniquely and B = – A.

Proof: Let A = $[a_{ij}]_{m\times n}$, since B = – A, so B = $[-\, a_{ij}]_{m\times n}$

A + B = $[a_{ij}]_{m\times n} + [-\, a_{ij}]_{m\times n}$
= $[a_{ij} + (-\, a_{ij})]_{m\times n}$ (definition the addition of matrices)
= $[a_{ij} - a_{ij}]_{m\times n}$
= $O_{m\times n}$

B + A = $[-\, a_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$
= $[(-\, a_{ij}) + a_{ij}]_{m\times n}$ (definition the addition of matrices)
= $[-\, a_{ij} + a_{ij}]_{m\times n}$
= $O_{m\times n}$

Theorem: Let A, B and C are three matrices of the same degree, then

(1) $A + B = A + C \Leftrightarrow B = C$

(2) $B + A = C + A \Leftrightarrow B = C$

Proof (1):

$A + B = A + C \Leftrightarrow -A + (A + B) = -A + (A + C)$ (add (–A) to each side from left)

$\Leftrightarrow (-A + A) + B = (-A + A) + C$ (the addition of matrices is associative)

$\Leftrightarrow O + B = O + C$

$\Leftrightarrow B = C$

Proof (2) (Home Work)

Subtraction of Matrices: Matrices are said to be compatible with subtraction if and only if they have the same degree.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices. The subtraction of A and B is denoted by A – B is also an m × n matrix, and

$$\begin{aligned} A - B &= A + (-B) \\ &= [a_{ij}]_{m\times n} + [-b_{ij}]_{m\times n} \\ &= [a_{ij} - b_{ij}]_{m\times n} \\ &= [c_{ij}]_{m\times n} \quad \text{where} \quad a_{ij} - b_{ij} = c_{ij} \text{ for all possible values of i, j} \end{aligned}$$

Examples:

(1) $\begin{bmatrix} 8 & 6 & -4 \\ 1 & 10 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 8-2 & 6-1 & -4-(-1) \\ 1-3 & 10-0 & -1-(-2) \end{bmatrix} = \begin{bmatrix} 6 & 5 & -3 \\ -2 & 10 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$ not define since the matrices have different size.

(3) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & \frac{1}{3} & -1 \\ \frac{1}{2} & 0 & -7 \end{bmatrix}$, find A + 2 B , A – 2B

$$2B = 2\begin{bmatrix} 2 & \frac{1}{3} & -1 \\ \frac{1}{2} & 0 & -7 \end{bmatrix} = \begin{bmatrix} 4 & \frac{2}{3} & -2 \\ 1 & 0 & -14 \end{bmatrix}$$

(4) If both matrices A and B are square matrix of the same order with real entries, then it is not necessary that $(AB)^2 = A^2 B^2$.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -2 & -2 \end{bmatrix}$$

$$(AB)^2 = (AB)\,(AB) = \begin{bmatrix} -2 & 0 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 8 & 4 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

$$B^2 = BB = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A^2 B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -4 & 4 \end{bmatrix}$$

We get that $(AB)^2 \neq A^2 B^2$.

Note that this relation is true when the matrices are commutative, for example consider $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -3 \\ -2 & 2 \end{bmatrix}$ apply that $(AB)^2 = A^2 B^2$ **(Home work)**

Theorem: Let A be a matrix of degree m×n, then

(1) $A I_n = A$

(2) $I_m A = A$

(3) $AO = O$, $OA = O$

Proof (1): Let $A = [a_{ij}]_{m\times n}$, $I_n = [s_{ij}]_{n\times n}$ such that $s_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

(i,j) element of $AI_n = \sum_{k=1}^{n} a_{ik} s_{kj}$

$= a_{i1}s_{1j} + a_{i2}s_{2j} + \ldots + a_{ij}s_{jj} + \ldots + a_{in}s_{nj}$

$= a_{i1}(0) + a_{i2}(0) + \ldots + a_{ij}(1) + \ldots + a_{in}(0)$

$= a_{ij}(1)$

$= a_{ij}$

= (i,j) element of A

Degree of the matrix $A = m \times n$ = Degree of the matrix $A_{m\times n} I_{n\times n}$, so $A I_n = A$.

Proof (2): **(Home work)**

Proof (3): Let $A = [a_{ij}]_{m\times n}$, $O = [f_{ij}]_{n\times p}$ such that $f_{ij} = 0$ for all i and j

$$A_{m\times n}\, O_{n\times p} = \left[\sum_{k=1}^{n} a_{ik} f_{kj}\right]_{m\times p}$$

Since $f_{ij} = 0$ for all i and j , then $a_{ik} f_{kj} = 0$. So

$A_{m\times n}\, O_{n\times p} = O_{m\times p}$

Degree of the matrix A O = m×p = Degree of the matrix O

$\therefore$ AO = O

In the same way we can prove that OA = O.

Theorem: Associative law of multiplication

Let A, B and C matrices compatible with multiplication, then

(1) (AB)C = A(BC)

(2) r(AB) = (rA)B = A(rB), where r is a real number and $A, B \in M_{n\times n}(F)$, $F = \mathbb{R}$ or $\mathbb{C}$.

Proof (1): Let $A = [a_{ij}]_{m\times p}$, $B = [b_{jk}]_{p\times q}$ and $C = [c_{ks}]_{q\times n}$

$(AB)\, C = ([a_{ij}]_{m\times p}\, [b_{jk}]_{p\times q})\, [c_{ks}]_{q\times n}$

$$= \left[\sum_{j=1}^{p} a_{ij} b_{jk}\right]_{m\times q} [c_{ks}]_{q\times n} \quad \text{(definition the multiplication of matrices)}$$

$$= \left[\sum_{k=1}^{q} \sum_{j=1}^{p} (a_{ij} b_{jk}) c_{ks}\right]_{m\times n} \quad \text{(definition the multiplication of matrices)}$$

$$= \left[\sum_{j=1}^{p} \sum_{k=1}^{q} a_{ij} (b_{jk} c_{ks})\right]_{m\times n} \quad \text{(the multiplication of numbers is associative)}$$

$A(BC) = [a_{ij}]_{m\times p}\, ([b_{jk}]_{p\times q}\, [c_{ks}]_{q\times n})$

$$= [a_{ij}]_{m\times p} \left[\sum_{k=1}^{q} b_{jk} c_{ks}\right]_{p\times n} \quad \text{(definition the multiplication of matrices)}$$

$$= \left[\sum_{j=1}^{p} \sum_{k=1}^{q} a_{ij} (b_{jk} c_{ks})\right]_{m\times n} \quad \text{(definition the multiplication of matrices)}$$

$\therefore$ (AB) C = A(BC)

Proof (2): Let A = $[a_{ij}]_{n\times n}$, B = $[b_{ij}]_{n\times n}$ and C = $[c_{ij}]_{n\times n}$, r is a real number

r(AB) = r $([a_{ij}]_{n\times n}[b_{ij}]_{n\times n})$

$= r\,[c_{ij}]_{n\times n}$ (definition the multiplication of matrices)

$= [r\,c_{ij}]_{n\times n}$ (definition the multiplication of matrix by scalar)

$= \left[r\sum_{k=1}^{n} a_{ik} b_{kj} \right]_{n\times n}$ (definition the multiplication of matrices)

$= \left[\sum_{k=1}^{n} r(a_{ik} b_{kj}) \right]_{n\times n}$

$= \left[\sum_{k=1}^{n} (r a_{ik}) b_{kj} \right]_{n\times n}$ (the multiplication of numbers is associative)

$= [r\,a_{ij}]_{n\times n}\,[b_{ij}]_{n\times n}$ (definition the multiplication of matrices)

$= (r\,[a_{ij}]_{n\times n})\,[b_{ij}]_{n\times n}$ (definition the multiplication of matrix by scalar)

= (rA)B

As the same way we can prove

r(AB) = A(rB) and (rA)B = A(rB) **(Home work)**

Theorem: Distributive law of multiplication over addition

Let A, B and C matrices compatible with multiplication, then

(1) A(B + C) = AB + AC

(2) (B + C) A = BA + CA

Proof (1): Let A = $[a_{ij}]_{m\times n}$, B = $[b_{jk}]_{n\times p}$, C = $[c_{ks}]_{n\times p}$.

Suppose that B + C = D such that $[b_{jk}]_{n\times p} + [c_{ks}]_{n\times p} = [b_{jk} + c_{ks}]_{n\times p} = [d_{ij}]_{n\times p}$

A (B + C) = A · D

$= \left[\sum_{k=1}^{n} a_{ik} d_{kj} \right]_{m\times p}$ (definition the multiplication of matrices)

$= \left[\sum_{k=1}^{n} a_{ik} (b_{kj} + c_{kj}) \right]_{m\times p}$ $d_{kj} = b_{kj} + c_{kj}$

$= \left[\sum_{k=1}^{n} (a_{ik} b_{kj} + a_{ik} c_{kj}) \right]_{m\times p}$ (distribution of multiplication over the addition in numbers)

$$A(B+C) = \left[\sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj} \right]_{m \times p}$$

$$= \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right]_{m \times p} + \left[\sum_{k=1}^{n} a_{ik} c_{kj} \right]_{m \times p} \qquad \text{(definition the addition of matrices)}$$

$$= AB + AC$$

$\therefore$ A (B + C) = AB + AC

Degree of the matrix B + C = n × p
Degree of the matrix A(B + C) = m × p
Degree of the matrix AB = m × p
Degree of the matrix AC = m × p
Degree of the matrix AB + AC = m × p

(A(B + C) = m × p and AB + AC = m × p: equal)

Proof (2): (Home work)

Definition: If A is any square matrix, then we can define

$$A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k \text{ times}}$$

Where k is any positive integer number.
Note that $A^0 = I$.

Theorem: If A is any square matrix, then for any positive integer numbers s and t
(1) $A^s \cdot A^t = A^{s+t}$
(2) $(A^s)^t = A^{st}$
We use the mathematical induction method in the proof.

Proof (1):
When t = 1 $\Rightarrow$ $A^s \cdot A^1 = A^{s+1}$
Suppose that the statement is true when t = k $\Rightarrow$ $A^s \cdot A^k = A^{s+k}$
Is the statement still true when t = k + 1?
i.e. $A^s \cdot A^{k+1} = A^{s+k+1}$

$$A^s \cdot A^{k+1} = A^s (A^k \cdot A)$$
$$= (A^s \cdot A^k) \cdot A \qquad \text{(the multiplication of matrices is associative)}$$
$$= A^{s+k} \cdot A$$
$$= A^{s+k+1}$$

Proof (2):

When $t = 1 \Rightarrow (A^s)^1 = A^s$

Suppose that the statement is true when $t = k \Rightarrow (A^s)^k = A^{s\,k}$

Is the statement still true when $t = k + 1$?

i.e. $(A^s)^{k+1} = A^{s\,(k+1)}$

$$\begin{aligned}(A^s)^{k+1} &= (A^s)^k \cdot (A^s)^1 \\ &= A^{sk} \cdot A^s \\ &= A^{sk+s} \\ &= A^{s\,(k+1)}\end{aligned}$$

Example: Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, compute A^2, A^3 and find A^k for any positive integer number.

Solution:

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^4 = A^3 \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^5 = A^4 \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\vdots \qquad \vdots \qquad\qquad\qquad\qquad \vdots$$

Note that:

When $k = 2n$ (even positive integer number)

$$A^2 = A^4 = \ldots = A^{k=2n} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

When $k = 2n - 1$ (odd positive integer number)

$$A = A^3 = \ldots = A^{k=2n-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Theorem: For any matrices A, B and C,

(1) $(A^t)^t = A$ (the transpose of transpose matrix is equal to the matrix itself)

(2) $(A \pm B)^t = A^t \pm B^t$

(3) $(\alpha A)^t = \alpha A^t$, where α is standard number

(4) $O^t_{m\times n} = O_{n\times m}$

(5) $I^t_n = I_n$ (the transpose of identity matrix is equal to the identity matrix itself)

Proof (1): Let $A = [a_{ij}]_{m\times n}$

$A^t = [a_{ji}]_{n\times m}$ (definition of matrix transpose)

$(A^t)^t = [a_{ij}]_{m\times n}$ (definition of matrix transpose)

$\therefore (A^t)^t = A$

Proof (2): Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$

Suppose $A \pm B = C = [c_{ij}]_{m\times n}$

$(A \pm B)^t = C^t = [c_{ji}]_{n\times m}$ (definition of matrix transpose)

$= [a_{ji} \pm b_{ji}]_{n\times m}$ $c_{ji} = a_{ji} \pm b_{ji}$

$= [a_{ji}]_{n\times m} \pm [b_{ji}]_{n\times m}$ (definition the addition and subtraction of matrices)

$\therefore (A \pm B)^t = A^t \pm B^t$

Generalization: For any compatible matrices with addition $A_1, A_2, \ldots, A_n$

$$(A_1 + A_2 + \ldots + A_n)^t = A_1^t + A_2^t + \ldots + A_n^t$$

Proof (3): $A = [a_{ij}]_{m\times n}$

$\alpha A = \alpha\, [a_{ij}]_{m\times n}$

$= [\alpha\, a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$(\alpha A)^t = [\alpha\, a_{ji}]_{n\times m}$ (definition the matrix transpose)

$= \alpha\, [a_{ji}]_{n\times m}$ (definition the multiplication of matrix by scalar)

$= \alpha A^t$

$\therefore (\alpha A)^t = \alpha A^t$

Proof (4): Let $O = [f_{ij}]_{m\times n}$, where $f_{ij} = 0$ for all values of i and j

$O_{m\times n} = [f_{ij}]_{m\times n}$

By definition of matrix transpose, we get

$O^t_{m\times n} = [f_{ji}]_{n\times m}$, where $f_{ji} = 0$ for all values of i and j

$$\therefore\ O^t_{m\times n} = O_{n\times m}$$

Proof (5): Let $I_n = [s_{ij}]_{n\times n}$, such that $s_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$I_n = [s_{ij}]_{n\times n}$

$I^t_n = [s_{ji}]_{n\times n}$ (definition of matrix transpose)

$s_{ij} = s_{ji} = 1$ if $i = j$

$s_{ij} = s_{ji} = 0$ if $i \neq j$

$$\therefore\ I^t_n = I_n$$

Theorem: Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$, then $(AB)^t = B^tA^t$

Proof:

Let $A^t = C = [c_{ij}]_{n\times m} \longrightarrow (c_{ij} = a_{ji})$

$B^t = D = [d_{ij}]_{p\times n} \longrightarrow (d_{ij} = b_{ji})$

$AB = E = [e_{ij}]_{m\times p} \longrightarrow \left(e_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \right)$

(i,j) element of $(AB)^t$ = (j,i) element of $AB = e_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$

(i,j) element of B^tA^t = (i,j) element of DC

$$= \sum_{k=1}^{n} d_{ik} c_{kj}$$

$$= \sum_{k=1}^{n} b_{ki} a_{jk} \qquad \text{(replaced)}$$

$$= \sum_{k=1}^{n} a_{jk} b_{ki} \qquad \text{(the multiplication of numbers is commutative)}$$

Degree of the matrix $AB = m \times p$

Degree of the matrix $(AB)^t = p \times m$ ← equal

Degree of the matrix $B^tA^t = p \times m$ ← equal

$$\therefore (AB)^t = B^tA^t$$

Problems:

(1) Is the matrix A equal to the zero matrix if $A^3 = O$, where A is a matrix of order 3×3?

Solution: $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \neq O_{3\times 3}$, but

$$A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3\times 3}$$

(2) For any n×n matrices A, B and C. Prove that

(a) $-(-A) = A$

(b) $A(B - C) = AB - AC$

(c) $(A - B)C = AC - BC$

Proof (a): Let $A = [a_{ij}]_{n\times n} \Rightarrow -A = [-a_{ij}]_{n\times n} \Rightarrow -(-A) = -[-a_{ij}]_{n\times n} = [a_{ij}]_{n\times n} = A$

Proof (b): Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $C = [c_{ij}]_{n\times n}$

$A(B - C) = [a_{ij}]_{n\times n} ([b_{ij}]_{n\times n} - [c_{ij}]_{n\times n})$

$= [a_{ij}]_{n\times n} [d_{ij}]_{n\times n}$ (definition the subtraction of matrices) and $d_{ij} = b_{ij} - c_{ij}$

$= \sum_{k=1}^{n} a_{ik} d_{kj}$ (definition the multiplication of matrices)

$= \sum_{k=1}^{n} a_{ik} (b_{kj} - c_{kj})$ $d_{kj} = b_{kj} - c_{kj}$

$= \sum_{k=1}^{n} (a_{ik} b_{kj} - a_{ik} c_{kj})$ (distribution the multiplication over the addition in numbers)

$= \sum_{k=1}^{n} a_{ik} b_{kj} - \sum_{k=1}^{n} a_{ik} c_{kj}$

$= [a_{ij}]_{n\times n} [b_{ij}]_{n\times n} - [a_{ij}]_{n\times n} [c_{ij}]_{n\times n}$ (definition the multiplication of matrices)

$= AB - AC$

Proof (c): **Home work**

(3) Let A and B be n×n matrices such that AB = BA. Prove that

(a) For any positive integer k, $AB^k = B^k A$.

(b) $(A + B)^2 = A^2 + 2AB + B^2$

Proof (a): If k = 1, then AB = BA by hypothesis

Suppose that the statement true for k = n, i.e. $AB^n = B^n A$

To prove the statement true when k = n + 1, i.e. to prove $AB^{n+1} = B^{n+1} A$

$AB^{n+1} = A(B^nB)$ (by pervious theorem $A^s A^t = A^{s+t}$)

$= (AB^n)B$ (the multiplication of matrices is associative)

$= (B^nA)B$ ($AB^n = B^nA$)

$= B^n(AB)$ (the multiplication of matrices is associative)

$= B^n(BA)$ ($AB = BA$)

$= (B^nB)A$ (the multiplication of matrices is associative)

$= B^{n+1}A$ (by pervious theorem $A^s A^t = A^{s+t}$)

Proof (B):

$(A + B)^2 = (A + B)(A + B)$ (definition the power of the matrices)

$= AA + AB + BA + BB$

$= A^2 + AB + AB + B^2$ $AB = BA$

$= A^2 + 2AB + B^2$

(4) Find all matrices $B \in M_{2\times2}(\mathbb{R})$ such that B commutes (commutative) with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Solution: Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a, b, c, d \in \mathbb{R}$

$AB = BA \Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} \Rightarrow b = 0\ ,\ a = d$

$\left\{\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}, a, c \in \mathbb{R}\right\}$.

(5) Let $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ and the polynomials $f(x) = x^2 + 3x - 10$ and $g(x) = x^2 + 2x - 11$.

Find the values of each polynomial? Is the matrix A is a root of each polynomial?

Solution:

$$f(A) = \left(\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\right)^2 + 3\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 12 & -9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 12 & -9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & -2 \end{bmatrix}$$

Hence, A is not a root of the polynomial $f(x)$.