Definition1.21: Let p and q are two propositions, then

1. The proposition "$q \to p$" is called the **converse** of "$p \to q$".
2. The proposition "$\sim p \to \sim q$" is called the **inverse** of "$p \to q$".

Example1.22: What is the converse and the inverse of the conditional statement:

"**if** $x > 5, x \in N$ **then** $x > 3$"?

What is the truth value of the statement and its inverse and converse?

Solution: The statement "**if** $x > 5$ **then** $x > 3$" Type equation here.

The truth value: T → T=T

The **converse** is "**if** $x > 3$ **then** $x > 5$"

The truth value: for $x = \{4,5\}$; T → F= F

For $x = \{6,7, \dots\}$, }; T → T= T

The **inverse** is **if** $x \leq 5$ **then** $x \leq 3$"

The truth value: for $x = \{1,2,3\}$; T → T= T

For $x = \{4,5\}$, }; T → F= F

Properties of the conditional operator: (→ (خواص أداة الشرط

Let p, q and r are three propositions. Using the **truth table** show that: (**H. W**) 1. $p \to q \neq q \to p$

2. $(p \to q) \to r \neq p \to (q \to r)$

3. **Find** the truth value of: $p \rightarrow T,\ p \rightarrow F,\ p \rightarrow \sim p,\ p \rightarrow p$

4. **Bi-conditional operator أداة الشرط المزدوج** : English word (if and only if), Arabic word (إذا وفقط إذا), symbol ($\leftrightarrow$)

Let p and q be propositions. The *bi-conditional* statement "$p \leftrightarrow q$" is the proposition "p if and only if q". The bi-conditional statement is **true** when p and q have the same true value, and is **false** otherwise.

لتكن كل من **p** و **q** عبارة بسيطة. تكون العبارة المركبة "**p** إذا وفقط إذا **q**" والتي يرمز لها بالرمز ($\leftrightarrow$) صادقة في حالة تشابه قيم صدق العبارتين وكاذبة فيما عدا ذلك.

P	Q	P if and only if Q
T	T	T
T	F	F
F	T	F
F	F	T

The truth table for the bi-conditional of two propositions

<u>**Remark1.23**</u>: There are some other ways to express "$p \leftrightarrow q$":

"p iff q"

"if p then q , and if q then p"

"p is necessary and sufficient for q".

<u>Example1.24</u>: Find the truth value of "$x > 0 \leftrightarrow 2x > 0$"

<u>Solution</u>: The statement is true because

If $x > 0$ then $2x > 0$ and if $2x > 0$ then $x > 0$

<u>Example1.25</u>: Find the truth value of "$x > 0 \leftrightarrow x^2 > 0$" **(H. W.)**

<u>Example1.26</u>: Let p: you can take the flight (True)

q: you can buy a ticket (True)

Then the bi-conditional statement $p \leftrightarrow q$ is

"you can take the flight if and only if you can buy a ticket"

Discuss the truth values of the bi-conditional statement.

Solution: The statement $p \leftrightarrow q$ is **true**

" you buy a ticket ↔ can take the flight"

or

"you do not buy a ticket ↔ cannot take the flight".

The statement $p \leftrightarrow q$ is **false** when p and q have opposite truth values.

"you do not buy a ticket ↔ you can take the flight"

or

"you buy a ticket ↔ cannot take the flight".

Properties of the biconditional operator خواص أداة الشرط المزدوج

Let p, q and r are three propositions. Using the **truth table** show that: **(H. W)**

1. $p \leftrightarrow q = q \leftrightarrow p$
2. $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$
3. **Find** the truth value of: $p \leftrightarrow T, p \leftrightarrow F, p \leftrightarrow \sim p, p \leftrightarrow p$.

Exercise1.27:

1. Find the truth value of the following statements:

[(if $2 + 3 = 4$ then $x + 4 = 4 + x$) and 8 is an even number] iff ($2 \leq -10$ or $2 \geq -10$).

Solution: $[(F \rightarrow T) \wedge T] \leftrightarrow (F \vee T) = [T \wedge T] \leftrightarrow T = T \leftrightarrow T = T$

2. Let p: horse can swim

q: Conjunction operator is useful

r: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ $for\ x, y \in N$

Find the truth value of the following statements:

1. $p \leftrightarrow r$
2. $(p \rightarrow r) \wedge q$
3. $[(p \rightarrow r) \vee (q \rightarrow \sim p)]$

3. Write the truth table of the following statements:

i) $\sim p \wedge q$
ii)$(p \wedge q) \rightarrow (p \vee q)$ **iii)** $(p \rightarrow q) \vee \sim (q \leftrightarrow p)$

4. Write the following statements using the connections operators $\rightarrow, \leftrightarrow, \wedge, \vee$

i) If p and q integer numbers and $q \neq 0$ then $\frac{p}{q}$ is a rational number

ii)If $x^2$is integer number then x is even or odd number **iii)** $xy > 0$ if and only if $(x > 0$ and $y > 0)$ or $(x < 0$ and $y < 0)$

Definition1.28: A compound proposition that is always true is called a ***tautology*** or ***lemma*** or ***theorem.***

يقال للعبارة المركبة التي تكون صادقة دائما بأنها **نظرية** أو **تحصيل حاصل**

A compound proposition that is always false is called a ***contradiction***.

يقال للعبارة المركبة والتي تكون خاطئة دائما بأنها تناقض.

Example1.29: Show that (p v ~ p) is tautology and ($p \wedge \sim p$) is contradiction.

Solution:

p	~ p	p V ~p tautology	p Λ ~ p contradiction
T	F	T	F
F	T	T	F

Example1.30: (H. W) which of the following compound statements is theorem (tautology) and which one is contradiction p Λ F , p V T , $p \leftrightarrow \sim p$, $[(p \rightarrow q) \wedge p] \wedge \sim q$

Definition1.31: Logical Equivalence التكافؤ المنطقي

Two statements (propositions) that have **same** truth values are called **logically equivalent**. The notation $p \equiv q$ or $p = q$ denotes that p and q are logically equivalent.

تكون عبارتين متكافئة منطقيا إذا كان لهما نفس قيمة الصدق ويرمز للتكافؤ المنطقي بالرمز ≡ أو =

Example1.32: show that ~ (p V q) = ~ p Λ ~ q (logically equivalent).

Hint: make truth table

Solution: The truth table for ~(p V q) and ~ p Λ ~ q is

P	Q	p ∨ q	~(p ∨ q)	~ p	~ q	~p ∧ ~q
T	T	T	**F**	F	F	**F**
T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

<u>Definition</u>1.33: Let p, q and r three propositions, then define the following logical equivalence:

1. $p \wedge q =$ $\sim(\sim p \vee \sim q)$
2. $p \rightarrow q =$ $\sim p \vee q$
3. $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

ملاحظة: التعريف أعلاه مهم جدا ويجب أن يحفظ

<u>De Morgan's Theorem:</u> Let p and q are two propositions. Then

1. $\sim(p \wedge q) = \sim p \vee \sim q$ **(H. W)**
2. $\sim(p \vee q) = \sim p \wedge \sim q$

<u>Proof (2)</u>: Take the **right hand side (R. H. S)**

$\sim p \wedge \sim q = \sim(\sim\sim \quad \vee \sim\sim \quad p\ q)$ [definition of $\wedge$]

$= \sim (p \vee q)$ [double negation law: $\sim\sim p=p$]

= **Left hand side (L. H. S).**

<u>**Exercise1.34:**</u> Simplify the following statements:

1. $\sim(p \vee \sim q)$

2. $\sim(\sim p \rightarrow q)$

3. $\sim(\sim p \leftrightarrow q)$

<u>**Solution**</u>**(1):** $\sim(p \vee \sim q) = \sim p \wedge \sim\sim q$ [De Morgan's law]

$= \sim p \wedge q$ $[\sim\sim q = q]$

<u>**Solution**</u>**(2):** $\sim(\sim p \rightarrow q) = \sim(\sim\sim p \vee q)$

$= \sim(p \vee q)$ $[\sim\sim p = p]$

$= \sim p \wedge \sim q$ [De Morgan's law]

<u>**Laws of Logical Equivalence**</u> **قوانين التطابق المنطقي**

Let p, q and r are propositions. The following are some of the common logical equivalence rules:

1. **Commutative Law قانون الإبدال:** $p \wedge q = q \wedge p$

$p \vee q = q \vee p$

$p \leftrightarrow q = q \leftrightarrow p$

2. **Associative Law قانون التجميع:** $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

$$(p \lor q) \lor r = p \lor (q \lor r)$$

$$(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$$

3. **Distributive Law (from left)** قانون التوزيع من اليسار:

$$p \land (q \lor r) = (p \land q) \lor (p \land r)$$

$$p \land (q \land r) = (p \land q) \land (p \land r)$$

$$p \lor (q \land r) = (p \lor q) \land (p \lor r)$$

$$p \lor (q \lor r) = (p \lor q) \lor (p \lor r)$$

$$p \lor (q \rightarrow r) = (p \lor q) \rightarrow (p \lor r)$$

$$p \lor (q \leftrightarrow r) = (p \lor q) \leftrightarrow (p \lor r)$$

4. **Distributive Law (from right)** قانون التوزيع من اليمين:

$$(q \lor r) \land p = (q \land p) \lor (r \land p)$$

$$(q \land r) \land p = (q \land p) \land (r \land p)$$

$$(q \land r) \lor p = (q \lor p) \land (r \lor p)$$

$$(q \lor r) \lor p = (q \lor p) \lor (r \lor p)$$

$$(q \rightarrow r) \lor p = (q \lor p) \rightarrow (r \lor p)$$

$$(q \leftrightarrow r) \lor p = (q \lor p) \leftrightarrow (r \lor p)$$

5. **Idempotent Law** قانون تساوي القوى : $p \land p = p;\ p \lor p = p$

6. **Identity Law:** $p \wedge T = p \,;\, p \vee F = p$

7. **Domination Law:** $p \wedge F = F; p \vee T = T$

<u>Exercise1.35:</u> Simplify the following statements using laws of logical equivalence:

بسط العبارات التالية باستخدام قوانين التطابق المنطقي

1. $(p \vee q) \wedge \sim p$

2. $(p \vee q) \vee (\sim p \wedge q)$

<u>Exercise1.36:</u> Prove (**without using the truth table**) that

برهن بدون استخدام جداول الصدق

$$\sim (p \vee (\sim p \wedge q)) = \sim p \wedge \sim q$$

<u>Solution</u>: Take the **L. H. S**

$$\sim (p \vee (\sim p \wedge q)) = \sim p \wedge \sim (\sim p \wedge q) \quad \text{[De Morgan's law]}$$

$$= \sim p \wedge (\sim\sim p \vee \sim q) \quad \text{[De Morgan's law]}$$

$$= \sim p \wedge (p \vee \sim q) \quad \text{[by double negation law]}$$

$$= (\sim p \wedge p) \vee (\sim p \wedge \sim q) \quad \text{[by distributive law]}$$

$$= F \vee (\sim p \wedge \sim q) \quad [\sim p \wedge p = F]$$

$$= (\sim p \wedge \sim q) \vee F \quad \text{[by commutative law]}$$

$$= \sim p \wedge \sim q \quad \textbf{R. H. S}$$

<u>Theorem1.37</u>: (Properties of →)

Let p, q and r are three propositions. Prove the following properties **without using truth tables:**

1. $p \rightarrow p = T$
2. $\sim p \rightarrow p = p$
3. $p \rightarrow T = T$
4. $T \rightarrow p = p$
5. $p \rightarrow F = \sim p$
6. $F \rightarrow p = T$
7. $p \rightarrow q = \sim q \rightarrow \sim p$
8. $p \rightarrow q = (p \wedge \sim q) \rightarrow \sim p$
9. $p \rightarrow q = (p \wedge \sim q) \rightarrow (r \wedge \sim r)$
10. $\sim (p \rightarrow q) = p \wedge \sim q$

<u>**Proof 1**</u>: To prove $p \rightarrow p = T$

$$p \rightarrow p = \sim p \vee p \quad \text{[def. of } \rightarrow\text{]}$$
$$= T$$

<u>**Proof 4**</u>: To prove $T \rightarrow p = p$

$$T \rightarrow p = \sim T \vee p \quad \text{[def. of } \rightarrow\text{]}$$
$$= F \vee p \quad [\sim T = F]$$
$$= p$$

Proof 7: To prove $p \to q = \sim q \to \sim p$

$$p \to q = \sim p \vee q \quad [\text{def. of } \to]$$

$$= q \vee \sim p \quad [\vee \text{ is commutative}]$$

$$= \sim q \to \sim p$$

Proof 8: To prove $p \to q = (p \wedge \sim q) \to \sim p$

Take the R. H. S: $(p \wedge \sim q) \to \sim p$

$$= \sim(p \wedge \sim q) \vee \sim p \quad [\text{def. of } \to]$$

$$= (\sim p \vee \sim\sim q) \vee \sim p \quad [\text{De Morgan}]$$

$$= (\sim p \vee q) \vee \sim p \quad [\sim\sim q = q]$$

$$= \sim p \vee (q \vee \sim p) \quad [\vee \text{ is associative}]$$

$$= \sim p \vee (\sim p \vee q) \quad [\vee \text{ is comm.}]$$

$$= (\sim p \vee \sim p) \vee q \quad [\vee \text{ is asso.}]$$

$$= \sim p \vee q \quad [p \vee p = p]$$

$$= p \to q \quad [\text{def. of } \to]$$

= L. H. S

Theorem1.38: (Properties of $\leftrightarrow$)

Let p and q are two propositions. Prove the following properties without using truth tables:

1. $p \leftrightarrow p = T, \; p \leftrightarrow T = p, \; p \leftrightarrow F = \sim p$

2. $p \leftrightarrow \sim p = F$

3. $\sim p \quad q = \sim q \quad p$

4. $p \leftrightarrow \sim p \leftrightarrow \sim q = q$

5. $\sim p \leftrightarrow q = p \leftrightarrow \sim q$

6. $\sim (p \leftrightarrow q) = \sim p \leftrightarrow q$

7. $\sim (p \leftrightarrow q) = p \leftrightarrow \sim q$

Proof 1: To prove $p \leftrightarrow T = p$

$$p \leftrightarrow T = (p \rightarrow T) \wedge (T \rightarrow p) \quad \text{[def. of } \leftrightarrow \text{]}$$

$$= (\sim p \vee T) \wedge (\sim T \vee p) \quad \text{[def. of } \rightarrow \text{]}$$

$$= (\sim p \vee T) \wedge (F \vee p) \quad [\sim T = F]$$

$$= T \wedge p \quad [\sim p \vee T = T]$$

$$= p$$

Proof 6: $\sim (p \leftrightarrow q) = \sim p \leftrightarrow q$

Take **L. H. S :** $\sim (p \leftrightarrow q) = \sim [(p \rightarrow q) \wedge (q \rightarrow p)]$ [def. of $\leftrightarrow$]

$$= \sim (p \rightarrow q) \vee \sim (q \rightarrow p) \quad \text{[De Morgan]}$$

$$= \sim (\sim p \vee q) \vee \sim (\sim q \vee p) \quad \text{[def. of } \rightarrow \text{]}$$

$$= (p \wedge \sim q) \vee (q \wedge \sim p) \quad \text{[De Morgan]}$$

$$= [(p \wedge \sim q) \vee q] \wedge [(p \wedge \sim q) \vee \sim p)] \quad \text{[distributive (v on } \wedge \text{)]}$$

$$= [(p \vee q) \wedge (\sim q \vee q)] \wedge [(p \vee \sim p) \wedge (\sim q \vee \sim p)] \quad \text{[dist. (v on } \wedge \text{)]}$$

$= [(p \vee q) \wedge T] \wedge [T \wedge (\sim q \vee \sim p)]$

$= (p \vee q) \wedge (\sim q \vee \sim p)$

$= (\sim p \rightarrow q) \wedge (q \rightarrow \sim p)$ [def. of $\rightarrow$] $= \sim p \leftrightarrow q$ [def. of $\leftrightarrow$]

Mathematical Proof البرهان الرياضي

A mathematical proof is a valid argument that establishes the truth of a mathematical statement.

البرهان الرياضي هو إثبات صحة عبارة رياضية من خلال حجة أو تعليل منطقي.

Methods of Proving Mathematical Statements (or Theorems)

1. Direct Proof of a conditional statement $p \rightarrow q$

Direct proofs lead from the hypothesis of a theorem to the conclusion.

Definition1.39: The integer number x is called **even** if there exist $k \in Z$ such that $x = 2k$.

Definition1.40: The integer number x is called **odd** if there exist $k \in Z$ such that $x = 2k + 1$.

Theorem1.41: If x is an odd natural number ($x \in O$) then x^2 is odd

Proof: Assume that x is an odd natural number. We must prove x^2 is odd

Since x is odd, then $x = 2k + 1$ for some $k \in N$.

$x^2 = x.x = (2k + 1)(2k + 1) = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Let $s = 2k^2 + 2k \in N$, then $x^2 = 2s + 1$

Hence, x^2 is an odd number.

Theorem1.42: (H. W.) If x is an even natural number ($x \in$ E) then x^2 is even.

Theorem1.43: The sum of two even natural numbers is even

The theorem can be written as follows: If x, y$\in E^+$ then $x + y \in E^+$ where $E^+ =$ set of positive even numbers.

Proof: Let p: x and y are even positive numbers,

q: $x + y$ is an even positive number

Let x = 2r and y = 2s $(r, s \in N)$. Then x + y = 2r + 2s = 2(r + s) such that $r + s \in N$

x + y= $2k$ where $k = r + s$. Therefore x + y is a positive even number.

Theorem1.44: (H. W.)

i) If $x \in E$ and $y \in O$ then $x + y \in O$

E +O = O

ii) If $x \in E$ and $y \in O$ then xy~ E iii)

E.O = E

iii) If $x, y \in E$ then $x + y \in E$

E + E = E

2. **Direct Proof of a conditional statement p ↔ q**

To prove a proposition in the form p ↔ q, we prove its equivalence. i.e.,

$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

Theorem1.45: x is odd number ↔ x+1 is an even

number **Proof:** Let p: x is odd number q: x+1 is

an even number

من التعريف أعلاه يجب أن نبرهن بان p→ q و q →p

1. **Prove p →q:** Let $x \in O$, $x = 2k + 1$; $k \in Z$

$x+1 = 2k + 2 = 2(k + 1)$; $(k + 1) \in Z$

$$x + 1 = 2r \quad ; r = k + 1 \in Z$$

$$x + 1 \in E$$

2. **Prove q → p:** Let $x + 1 \in E$ To prove $x \in O$

$$x + 1 = 2k \quad ; k \in Z$$

$$x = 2k - 1; \; k \in Z \; \ldots\ldots(1)$$

Since $k \in Z$, then $r = k - 1 \in Z$

$$k = r + 1 \ldots\ldots(2)$$

Substitute (2) in (1), $x = 2(r + 1) - 1 = 2r + 1$; $r \in Z$

$$x = 2r + 1 \in O$$

<u>**Theorem1.46**</u>: x is even ↔ x^2 is even

<u>**Proof:**</u> Let p: x is even number

q: x^2 is even number

من التعريف أعلاه يجب أن نبرهن بان $p \to q$ و $q \to p$

1. **Prove p →q:** Let $x \in E$, $x = 2k$; $k \in Z$

Prove $x^2 \in E$ (Theorem (1.44) مشابه لبرهان)

2. **Prove q → p:** Let $x^2 \in E$ To prove $x \in E$

Take $x^2 + x = x(x + 1) \in E$ [from Theorem 1.46(ii)]

$\Rightarrow x = x(x + 1) - x^2 \in E$ [Theorem 1.46(iii)]

$\Rightarrow x \in E$

E + E = E(0)

Theorem1.47: (H. W.) x is odd number if and only if x^2 is odd number.

3. Proof by Contradiction

البرهان بالتناقض هو أن نفرض عكس المطلوب إثباته ثم نحصل على تناقض مع الفرض

Theorem1.48: Prove that: If $x^2 \in O$ then $x \in O$

Proof: Assume that $x^2 \in O$. To prove $x \in O$

By contradiction, assume that $x \in E$

$$x = 2k \; ; k \in Z$$

$$x^2 = 4k^2 \in E$$

تناقض مع الفرض لأنه في الفرض $x^2 \in O$

$\therefore x \in O.$

Theorem1.49: If x^2 is even then x is even

Proof: Assume that $x^2 \in E$. To prove $x \in E$

By contradiction, assume that $x \in O$

$$x = 2k + 1 \; ; k \in Z$$

تناقض مع الفرض $x^2 = 4k^2 + 4K + 1 \in O$

$\Rightarrow x^2 \in E$. Hence, $x \in E$.

Theorem1.50: Prove that: If $n = ab$ where a and b are positive integers, then $a \leq \sqrt{n} \vee b \leq \sqrt{n}$.

Proof: Let p: $n = ab$ where a and b are positive integer **hypothesis**

q: $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ **conclusion**

The first step is to assume that the **conclusion** is false as follows:

Assume that $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ is false (F). Hence, $\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n})$ is true (T).

$\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n}) = \sim(a \leq \sqrt{n})$ and $\sim(b \leq \sqrt{n})$ [De Morgan's law]

$= a > \sqrt{n}$ and $b > \sqrt{n}$

Multiply the two inequalities together, $ab > n$ هذه المتراجحة تناقض الفرض

This shows that $ab \neq n$ **contradiction with the hypothesis**

Thus, $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ is true.

Definition1.51: Variable المتغير

An alphabetic letter $x, y, z, \ldots$ which represents a number that is either arbitrary or unknown.

Example1.52: "$4x - 7 = 5$": x is a variable

"$\sqrt[3]{z} = 3$": z is a variable

Definition1.53: Open Sentence الجملة المفتوحة

A sentence is called **open sentence** (or propositional function), if it contains one or more variables. Open sentence is denoted by p(x), q(x), g(x)…etc.

(الكلمة المفتوحة) اذا احتوت على متغير واحد او اكثر ونرمز لها بالرموز p(x), q(x) g(x) ...

Example1.54: The following are open sentences:

$p(x)$: x is an odd number

$q(x, y): x + y = 5$ such that $x, y \in N$

$r(z): \sqrt[3]{z} = 3$ such that $z \in R$

$s(y)$: computer y is working properly

Example1.55: Let the open sentence `` $p(x): x > 3$".

What are the truth value of $p(5)$ and $p(-1)$? Which values $x \in N$ that make $p(x)$ true?

Solution: $p(5): 5 > 3$ is a true proposition

$p(-1): -1 > 3$ is a false proposition

$p(x)$ is a true statement for $x \in \{4, 5, 6, \ldots\}$.

Example1.56: Let the propositional function $q(x, y): x = y + 3$. What are the truth values of $q(1,2)$ and $q(3,0)$?

Solution: $q(1,2): 1 = 2 + 3$. This means $1 = 5$ which is false. Thus, $q(1,2)$ is false statement

$q(3,0): 3 = 0 + 3 = 3$. Hence $q(3,0)$ is true proposition

Example1.57: (H. W) Let the open sentence ``$r(x, y, z): x + y = z$".

What are the truth values of $r(1,2,3)$ and $r(0,0,1)$?

Definition1.58: Solution Set (Truth set)

Let $p(x)$ be an open sentence and let A be a set. The solution set denoted by T_p is the set of all elements x of A for which $p(x)$ is true. In other words

$$T_p = \{x \in A : p(x) \text{ is true}\}$$

مجموعة الحل أو مجموعة الصدق: هي مجموعة العناصر التي تجعل التعبير المفتوح p(x) عبارة صادقة.

T_p