Complex analysis (lecture 22)

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- **Singular points and poles النقاط الشاذة والاقطاب**
- **Zeros of function اصفار الدالة**
- **Residus Theory نظرية الرواسب**

Singular points and poles النقاط الشاذة والاقطاب

Definition (22-1): Let z_0 be an isolated singular point of a function f and let $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be the Laurent series of f in the domain $0<|z-z_0|<R$ $\ni$

$$f(z)=\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

$$=\sum_{n=-\infty}^{-1} a_n(z-z_0)^n+\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

The series $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ is called principal part .

Example (22-1): Find the principal part of f at $z=1$ if

$$f(z)=\frac{2}{(z-1)^2(z-2)}$$

Solution : $z_0=1$

$$f(z)=\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n=\sum_{n=0}^{\infty} a_n(z-1)^n$$

$$\frac{1}{z-2}=\frac{1}{(z-1)-1}=\frac{-1}{1-(z-1)}=-\sum_{n=0}^{\infty}(z-1)^n$$

$$f(z)=\frac{2}{(z-1)^2(z-2)}$$

$$=\frac{-2}{(z-1)^2}\sum_{n=0}^{\infty}(z-1)^n \qquad ,0<|z-1|<1$$

$$f(z)=-2\sum_{n=0}^{\infty}(z-1)^{n-2}$$

$$= -2((z-1)^{-2} + (z-1)^{-1} + 1 + (z-1) + \cdots)$$

The principal part is $-2(z-1)^{-2} - 2(z-1)^{-1}$

Determine (classify) isolated singular points تصنيف النقاط الشاذة

[1] If $a_n = 0\ \forall n \leq -1\ then\ z_0$ is called a removable singular point نقطة شاذة قابلة للإزالة

[2] If there exists a negative integer $m \ni a_m \neq 0\ \ and\ a_n = 0$ for $n < m$ then z_0 is called a pole of order –m

Remark : if $m = 1$ then z_0 is called a simple pole

[3] If $a_n \neq 0\ \forall n \leq -1\ then\ z_0$ is called a essential singular point نقطة شاذة اساسية

Example (22-2): Find and classify the isolated singular points of

$$f(z) = \frac{e^{z^2}}{z^4}$$

Solution : f is analytic $\forall z \in \mathbb{C} - \{0\}$

$z = 0$ is an isolated singular point

$$f(z) = \frac{e^{z^2}}{z^4} = \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{z^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty}\frac{z^{2n-4}}{n!} \quad , 0 < |z| < \infty$$

$$f(z) = z^{-4} + z^{-2} + \frac{1}{2} + \frac{1}{6}z^2 + \cdots$$

$\therefore z = 0$ is a pole of order 4 .

Example (22-3): Find and classify the isolated singular points of

$$f(z) = \frac{\cos(2z) - 1}{z^2}$$

Solution : f is analytic $\forall z \in \mathbb{C} - \{0\}$

$z = 0$ is an isolated singular point

$$\cos(2z) = \sum_{n=0}^{\infty} \frac{(-1)^n (2z)^{2n}}{2n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2z)^{2n}}{2n!}$$

$$\cos(2z) - 1 = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (2z)^{2n}}{2n!} - 1$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n\, 4^n (z)^{2n}}{2n!}$$

$$f(z) = \frac{\cos(2z) - 1}{z^2} = \frac{1}{z^2} \sum_{n=1}^{\infty} \frac{(-1)^n\, 4^n (z)^{2n}}{2n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n\, 4^n (z)^{2n-2}}{2n!} \quad ,0 < |z| < \infty$$

$$f(z) = -2 + \frac{2}{3} z^2 - \frac{64}{6!} z^4 + \cdots$$

$\therefore z = 0$ is a removable singular point .

Example (22-4): Find isolated singular points of

$$f(z) = e^{-\frac{1}{z}}$$

and determine whether the are poles , removable or essential singular points .

Solution : f is analytic $\forall z \in \mathbb{C} - \{0\}$

$z = 0$ is an isolated singular point

$$f(z) = e^{-\frac{1}{z}}$$

$$= \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{z}\right)^n}{2n!} = \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n n!} \;, |z| > 0$$

$$f(z) = 1 - z^{-1} + \frac{1}{2} z^{-2} - \frac{1}{6} z^{-3} + \cdots$$

$\therefore z = 0$ is a essential singular point .

Example (22-5): Find and classify the isolated singular points of:

[1] $f(z) = \frac{z^2-9}{z-3}$

Solution :

$z = 3$ is an isolated singular point

$$f(z) = \frac{z^2 - 9}{z - 3} = \frac{(z-3)(z+3)}{z-3} = z + 3$$

$\therefore z = 3$ is a removable singular point .

[2] $f(z) = \frac{z^2+3z-4}{z+4}$

Solution :

$z = -4$ is an isolated singular point

$$f(z) = \frac{z^2 + 3z - 4}{z + 4} = \frac{(z+4)(z-1)}{z+4} = z - 1$$

$\therefore z = -4$ is a removable singular point .

Example (22-6): Find isolated singular points of

$f(z) = sin\frac{1}{z}$ and determine whether the are poles , removable or essential singular points (H.W)

Zeros of function اصفار الدالة

Definition (22-2): A zero of analytic function $f(z)$ in the domain D is z_0 such that $f(z_0) = 0$

Order of zero

- A zero has order n if $f(z_0), f'(z_0), f''(z_0), \ldots, f^{(n-1)}(z_0)$ Are all zero at $z = z_0$ but $f^{(n)}(z_0) \neq 0$
- If $f(z_0) = 0 \; and \; f'(z_0) \neq 0$ then f has a simple zero .

Example (22-7): Find the zero and their order for functions :

[1] $f(z) = z^2 + 1$

Solution :

$$f(z) = 0 \Rightarrow z^2 + 1 = 0 \Rightarrow z = \pm i$$

$$f'(z) = 2z \Rightarrow f'(i) = 2i \neq 0$$

$$f'(-i) = 2i \neq 0$$

$$\therefore f(z) = 0 \; and \; f'(z) \neq 0$$

$\therefore f$ has a simple zero .

[2] $f(z) = 2z(e^z - 1)$

Solution :

$$f(z) = 0 \Rightarrow 2z(e^z - 1) = 0 \Rightarrow z = 0$$

$$f'(z) = 2ze^z + 2\,(e^z - 1) \Rightarrow f'(0) = 0$$

$$f''(0) = 2ze^z + 4e^z = 4 \neq 0$$

$$\therefore f(z) = 0 \; and \; f''(z) \neq 0$$

$\therefore f$ has a zero of order 2 .

[3] $f(z) = (z^4 - 16)^2$

Solution :

$$f(z) = 0 \implies (z^4 - 16)^2 = 0 \implies (z^2 - 4)(z^2 + 4) = 0$$

$$z = \pm 2 \ \ and \ z = \pm 2i$$

$$f'(z) = 2\,(z^4 - 16)\,(4z^3) = 8z^3(z^4 - 16)$$

$$\implies f'(2) = 0 \ \ , f'(2i) = 0$$

$$f''(2) = 32z^6 + 24z^2(z^4 - 16)$$
$$f''(2) = 32(64) + 24(4)(16 - 16) = 32(64) \neq 0$$

$$\therefore f(z) = 0 \ and \ f''(z) \neq 0$$

$\therefore f$ has a zero of order 2 .

[4] $f(z) = (e^z + 1\,)^3$ (H.W)

[5] $f(z) = e^z - z - 1$

Solution :

$$f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - z - 1$$

$$f(z) = \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$f(z) = 0 \implies \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = 0 \implies z = 0$$

$$f'(z) = \frac{2z}{2!} + \frac{3z^2}{3!} + \cdots$$

$$\implies f'(0) = 0$$

$$f''(2) = \frac{2}{2!} + \frac{6z}{3!} + \cdots$$
$$f''(0) \neq 0$$

$$\therefore f(z) = 0 \; and \; f''(z) \neq 0$$

$\therefore f$ has a zero of order 2 .

[6] $f(z) = cotz$

Solution :

$$f(z) = \frac{cosz}{sinz}$$

$$f(z) = 0 \implies \frac{cosz}{sinz} = 0 \implies cosz = 0$$

$$z = \left(k + \frac{1}{2}\right)\pi, \quad k = 0, \pm1, \pm2, \ldots$$

$$f'\left(\left(k + \frac{1}{2}\right)\pi\right) = -sin\left(k + \frac{1}{2}\right)\pi \neq 0$$

$$\therefore f(z) = 0 \; and \; f'(z) \neq 0$$

$\therefore f$ has a simple zero .

[7] $f(z) = \tan z$ (H.W)

Poles الأقطاب

Example (22-8):

[1] $f(z) = \frac{1}{z-1}$ has a simple pole at $z = i$.

[2] $f(z) = \frac{cosz}{z}$ has a simple pole at $z = 0$

[3] $f(z) = \frac{3z}{z^2+1}$ has a simple pole at $z = \pm i$.

[4] $f(z) = z\frac{1}{z^2}$ has a pole of order 2 at $z = 0$

[5] $f(z) = \frac{1}{(z-3)^3}$ (H.W)

Residus Theory نظرية الرواسب

ايجاد راسب (باقي) الدالة عند النقطة المنفردة

$$Res(f,z_0)$$

Theorem (22-1): If f has a simple pole at z_0 then

$Res\ (f,z_0) = \lim_{z\to z_0}(z-z_0)f(z)$

Theorem (22-2): If f has a pole of order $\mathrm{n} > 1$ at z_0 then

$$Res\ (f,z_0) = \lim_{z\to z_0} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)$$

Example (22-9): Find $Res(f,z_0)$ of the function

$$f(z) = \frac{z+1}{z-i}$$

Solution : f has a simple pole at $z = i$

$$\therefore Res(f,z_0) = \lim_{z\to z_0}(z-z_0)f(z)$$

$$Res(f,i) = \lim_{z\to i}(z-i)\frac{z+1}{z-i}$$

$$Res(f,i) = \lim_{z\to i}(z+1) = 1+i$$

Example (22-10): Find $Res(f,z_0)$ of the function

$$f(z) = \frac{z+2}{z^2(z^2+1)}$$

Solution :

$$f(z) = \frac{z+2}{z^2(z+i)(z-i)}$$

[1] f has a simple pole at $z = i$

$$\therefore Res(f,z_0) = \lim_{z\to z_0}(z-z_0)f(z)$$

$$\therefore Res(f,i) = \lim_{z\to i}(z-i)\frac{z+2}{z^2(z+i)(z-i)}$$

$$Res(f,i) = \lim_{z\to i}\frac{z+2}{z^2(z+i)} = \frac{i+2}{-2i} = -\frac{1}{2}+i$$

[2] f has a simple pole at $z=-i$

$$\therefore Res(f,z_0) = \lim_{z\to z_0}(z-z_0)f(z)$$

$$\therefore Res(f,-i) = \lim_{z\to -i}(z+i)\frac{z+2}{z^2(z+i)(z-i)}$$

$$Res(f,i) = \lim_{z\to i}\frac{z+2}{z^2(z-i)} = \frac{-i+2}{2i} = -\frac{1}{2}-i$$

[3] f has a pole of order 2 at $z=0$

$$Res\ (f,z_0) = \lim_{z\to z_0}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z)$$

$$Res\ (f,0) = \lim_{z\to 0}\frac{d}{dz}\left[z^2\frac{z+2}{z^2(z+i)(z-i)}\right]$$

$$Res\ (f,0) = \lim_{z\to 0}\frac{d}{dz}\left[\frac{z+2}{(z+i)(z-i)}\right]$$

$$Res\ (f,0) = \lim_{z\to 0}\left[\frac{1-4z-z^2}{(z^2+1)}\right] = 1$$

Example (22-11): Find $Res(f,z_0)$ of the function (H.W)

$$f(z) = z^{-3}sinz$$

Theorem (22-3): If f,g two analytic function at

$z_0 \ni \mathrm{h}(z) = \frac{f(z)}{g(z)}$ then $Res\ (h,z_0) = \frac{f(z)}{g'(z)}$

Example (22-12): Find $Res(f, z_0)$ of the function

$$f(z) = tanz$$

Solution :

$$f(z) = \frac{sinz}{cosz}$$

$$cosz = 0 \implies z = \left(\frac{1}{2} + k\right)\pi\ , k = 0, \pm 1, \pm 2, \dots$$

$$g(z) = cosz \implies g'(z) = -sinz$$

$$Res\ (h, z_0) = \frac{f(z)}{g'(z)}$$

$$Res\ \left(h, \left(\frac{1}{2} + k\right)\pi\ \right) = \frac{sinz}{-sinz} = -1$$

Complex analysis (lecture 21)

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- **Taylor series** متسلسلة تايلر
- **Maclourin series** متسلسلة ماكلورين
- **Laurent series** مسلسلة لورنت

Taylor series متسلسلة تايلر

Definition (21-1): Suppose that f is analytic function of z_0 if then $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$

$$a_n=f(z_0)+f^{(1)}(z_0)+\frac{f^{(2)}(z_0)}{2!}+\frac{f^{(3)}(z_0)}{3!}+\cdots+\frac{f^{(n)}(z_0)}{n!}$$

$$a_n=\frac{f^{(n)}(z_0)}{n!}$$

If $z_0=0$ then the series is said to be Maclaurin .

Properties (21-1): The taylor series for

[1] $f(z)=e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$, $at\ z_0=0$

Solution : $f(0)=e^0=1$

$$f^{(1)}(z)=e^z \Longrightarrow f^{(1)}(0)=e^0=1$$

$$f^{(2)}(z)=e^z \Longrightarrow f^{(2)}(0)=e^0=1$$

$$\vdots$$

$$f^{(n)}(z)=e^z \Longrightarrow f^{(n)}(0)=e^0=1$$

$$a_n=\frac{f^{(n)}(z_0)}{n!}=\frac{1}{n!}$$

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{n!}(z-0)^n$$

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

[2] $f(z) = sinz = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$, $at\ z_0 = \pi$

[3] $f(z) = cosz = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$, $at\ z_0 = 0$

[4] $f(z) = coshz = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$

[5] $f(z) = sinhz = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

[6] $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

[7] $f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n$

Example (21-1): Find the taylor series for

[1] $f(z) = lnz$, $at\ z_0 = 1$

Solution : $f(0) = \ln(1) = 0$

$$f^{(1)}(z) = \frac{1}{z} \Longrightarrow f^{(1)}(1) = \frac{1}{1} = 1 = 0!$$

$$f^{(2)}(z) = -\frac{1}{z^2} \Longrightarrow f^{(2)}(1) = -\frac{1}{1} = -1 = -(1!)$$

$$f^{(3)}(z) = \frac{2}{z^3} \Longrightarrow f^{(3)}(1) = \frac{2}{1} = 2 = (2!)$$

$$f^{(4)}(z) = -\frac{6}{z^4} \Longrightarrow f^{(4)}(1) = -\frac{6}{1} = -6 = -(3!)$$

$$a_n = 0 + 0! - \frac{i!}{2!} + \frac{2!}{3!} - \frac{3!}{4!} + \cdots$$

$$= \frac{0!}{1!} - \frac{i!}{2!} + \frac{2!}{3!} - \frac{3!}{4!} + \cdots$$

$$a_n = (-1)^n \frac{(n-1)!}{n!} = \frac{(-1)^n}{n}$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (z-1)^n$$

[2] $f(z) = e^{2iz}$

Solution :

$$\because \; e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Longrightarrow e^{2iz} = \sum_{n=0}^{\infty} \frac{(2iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{(2i)^n z^n}{n!}$$

[3] $f(z) = e^{z^2}$

Solution :

$$\because \; e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Longrightarrow e^{z^2} = \sum_{n=0}^{\infty} \frac{(z)^{2n}}{n!}$$

[4] $f(z) = \sin(4z)$

Solution :

$$\because \; sinz = \sum_{n=0}^{\infty} \frac{(-1)^n \, z^{2n+1}}{(2n+1)!}$$

$$\Longrightarrow \sin(4z) = \sum_{n=0}^{\infty} \frac{(-1)^n \; (4z)^{2n+1}}{(2n+1)!}$$

[5] $f(z) = z^3 sin2z$

Solution :

$$\because\, sinz = \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2n+1}}{(2n+1)!}$$

$$z^3 sin2z = z^3 \sum_{n=0}^{\infty} \frac{(-1)^n\, (2z)^{2n+1}}{(2n+1)!}$$

$$= z^3 \sum_{n=0}^{\infty} \frac{(-1)^n\, (2\,)^{2n+1}}{(2n+1)!} (z\,)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, (2\,)^{2n+1}}{(2n+1)!} (z\,)^{2n+4}$$

[6] $f(z) = \cos z^2$ (H.W)

[7] $f(z) = \frac{1}{3-z}$

Solution :

$$\because\, \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$\Rightarrow\, \frac{1}{3-z} = \frac{1}{3\left(1-\frac{z}{3}\right)}$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}}$$

Laurent series مسلسلة لورنت

Definition (21-2): If f is analytic function $R_1 < |z - z_0| < R_2$ then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

$$= \sum_{n=-\infty}^{-1} a_n(z-z_0)^n + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

$$Let\ n = -k$$

$$= \sum_{k=\infty}^{1} a_{-k}(z-z_0)^{-k} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

$$= \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{k=1}^{\infty} \frac{a_k}{(z-z_0)^k}$$

Example (21-2): Find the Laurent series representation of

[1] $\sum_{n=0}^{\infty} z^{n-1} = z^{-1} + z^0 + z^1 + z^2 + z^3 + \cdots$

[2] $if\quad f(z) = e^{\frac{1}{z}}$

Solution :

$$\because\ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Longrightarrow e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{(\frac{1}{z})^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

[3] $f(z) = \frac{e^{2z}}{z^4}\quad ,0 < |z| < 1$

Solution :

$$z_0 = 0$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n z^n$$

$$\because\ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Longrightarrow e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$$

$$|z| < 1$$

$$f(z)=\frac{e^{2z}}{z^4}=\frac{1}{z^4}\sum_{n=0}^{\infty}\frac{2^n}{n!}z^n \quad 0<|z|<1$$

$$=\sum_{n=0}^{\infty}\frac{2^n}{n!}z^{n-4} \quad 0<|z|<1$$

[4] $f(z)=ze^{\frac{1}{z}}$ $\quad ,0<|z|<\infty$ (H.W)

[5] $f(z)=\frac{1}{z(1-z)}$ $\quad 0<|z|<1$

Solution :

$$z_0=0$$

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n=\sum_{n=0}^{\infty}a_nz^n$$

$$\because \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$$

$$|z|<1$$

$$f(z)=\frac{1}{z(1-z)}=\frac{1}{z}\sum_{n=0}^{\infty}z^n \quad 0<|z|<1$$

$$=\sum_{n=0}^{\infty}z^{n-1} \quad 0<|z|<1$$

Complex analysis (lecture 20)

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- **The sequence (المتتاليات) المتتابعات**
- **The series (المتسلسلة الهندسية) المتسلسلات**
- **The sum of the Geometric series مجموع المتسلسلة الهندسية**
- **Ratio test اختبار النسبة**

The sequence (المتتاليات) المتتابعات

Definition (20-1): Suppose $z_n \in \mathbb{C}$ *and* $z_0 \in \mathbb{C}$ then we say that $z_n \longrightarrow z_0$ if for each $\epsilon > 0\ \exists N$ *such that*

$n > N, |z_n - z_0| < \epsilon$ the sequence z_n is eventually contained in any $\epsilon - nbhd\ of\ z$

$$\lim_{n\to\infty} z_n = z_0 \Longrightarrow \forall\epsilon > 0\ \exists N \ni n > N \Longrightarrow |z_n - z_0| < \epsilon$$

Example(20-1):

[1] The sequence $\{i, -1, 1, i, \dots\}$

$z_n = (i)^n = \{i^n\}_{n=1}^{\infty}$

The domain its 1,2,3,4,5,…. and

The range its $i, i^2, i^3, i^4, i^5, \dots$

[2] The sequence $\{-1, -i, 1, i, \dots\}$

$z_n = (i)^{n+1} = \{i^{n+1}\}_{n=1}^{\infty}$

The domain its 1,2,3,4, …. and

The range its $i, i^2, i^3, i^4, i^5, \dots$

Example(20-2): Find the limit of the sequence

$$z_n = \left\{\frac{\sqrt{n} + i(n+1)}{n}\right\}_{n=1}^{\infty}$$

Solution : we write $z_n = x_n + iy_n$

$$z_n = \frac{1}{\sqrt{n}} + i\,\frac{n+1}{n}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$$

$$\therefore \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0 + i = i$$

Example(20-3): Show that the sequence

$\left(2-\frac{1}{n}\right) + i\left(5+\frac{1}{n}\right)$ its converges .

Solution :

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(2-\frac{1}{n}\right) = 2$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(5+\frac{1}{n}\right) = 5$$

$$\therefore \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 2 + 5i$$

$\therefore$ The sequence is converge

Example(20-4): Show that the sequence

$z_n = \left\{\frac{2n+i(n+2)}{n}\right\}_{n=1}^{\infty}$ its converges . (H.W)

Example(20-5): Show that the sequence

$$z_n = \{\,(1+i)^n\}_{n=1}^{\infty}$$

its diverges .

Solution :

$$r = \sqrt{1+1} = \sqrt{2}\ , tan^{-1}(1) = \frac{\pi}{4}$$

$$z_n = \left(\sqrt{2}\right)^n \left[cos\frac{n\pi}{4} + isin\frac{n\pi}{4}\right]$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\left(\sqrt{2}\right)^n cos\frac{n\pi}{4}\right) = \infty$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\left(\sqrt{2}\right)^n sin\frac{n\pi}{4}\right) = \infty$$

$$\therefore \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = \infty$$

$\therefore$ The sequence is diverges

The series (المتسلسلة الهندسية) المتسلسلات

Suppose $\{z_n\}$ is sequence then the complex numerical series

$$z_1 + z_2 + \cdots . + z_n = \sum_{n=1}^{\infty} z_n$$

Sequence of partial suns $\{S_n\}$ general term is

$$S_1 = z_n$$

$$S_2 = z_1 + z_2$$

$$\vdots$$

$$S_n = z_1 + z_2 + \cdots + z_n = \sum_{n=1}^{\infty} z_n$$

Example(20-6): Show that the series

$$\sum_{n=1}^{\infty} (1-i)^n$$

its diverges .

Solution :

$$r = \sqrt{1+1} = \sqrt{2} \ , tan^{-1}(1) = \frac{\pi}{4}$$

$$z_n = \left(\sqrt{2}\right)^n \left[cos\frac{n\pi}{4} + isin\frac{n\pi}{4}\right]$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\left(\sqrt{2}\right)^n cos\frac{n\pi}{4}\right) = \infty$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\left(\sqrt{2}\right)^n sin\frac{n\pi}{4}\right) = \infty$$

$$\therefore \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = \infty$$

$\therefore \sum_{n=1}^{\infty}(1-i)^n$ is diverges

<u>Definition (20-2):</u> Geometric series is a series with a constant between successive terms

$$S = \frac{a}{1-r} \quad ,r \neq 1$$

<u>Example(20-7):</u> Study the converges of

$$\sum_{n=1}^{\infty}\frac{2i}{3^n}$$

<u>Solution :</u> $S_n = 2i\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)$

$$a = \frac{2i}{3} \quad , \quad r = \frac{\frac{2i}{9}}{\frac{2i}{3}} = \frac{1}{3}$$

$$S = \frac{\frac{2i}{3}}{1-\frac{1}{3}} = \frac{2i}{2} = i$$

$$\therefore \sum_{n=1}^{\infty}\frac{2i}{3^n} \longrightarrow i$$

<u>Example(20-8):</u> Study the converges of

$$\sum_{n=1}^{\infty} \frac{2i}{(-1)^{n+1}(2^n)}$$

Solution : $S_n = 3i\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots\right)$

$$a = \frac{3i}{2} \quad , \quad r = \frac{\frac{-3i}{4}}{\frac{3i}{2}} = \frac{-1}{2}$$

$$S = \frac{\frac{3i}{2}}{1 + \frac{1}{2}} = \frac{3i}{3} = i$$

$$\therefore \sum_{n=1}^{\infty} \frac{2i}{(-1)^{n+1}(2^n)} \longrightarrow i$$

Example(20-9): Study the converges of (H.W)

$$\sum_{n=1}^{\infty} \left(\frac{1+2i}{1-i}\right)^n$$

The sum of the Geometric series مجموع المتسلسلة الهندسية

$$\sum_{n=0}^{\infty} z_n = \frac{1}{1-z}$$

$$if\ r < 1\ then \sum_{n=0}^{\infty} z_n\ is\ converges$$

$$if\ r > 1\ then \sum_{n=0}^{\infty} z_n\ is\ diverges$$

Example(20-10): Find the sum of

$$\sum_{n=0}^{\infty}(-1)^n \left(\frac{2}{3}\right)^n$$

Solution : Let $z = \frac{-2}{3}$

$$\sum_{n=0}^{\infty} z_n = \frac{1}{1-z} = \frac{1}{1+\frac{2}{3}} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$$

Example(20-11): Find the sum of

$$\sum_{n=1}^{\infty} \left(\frac{1}{2i}\right)^n$$

Solution : Let $z = \frac{1}{2i} \times \frac{-i}{-i} = \frac{-i}{2}$

$$\sum_{n=1}^{\infty} z_n = \frac{1}{1-z} = \frac{1}{1+\frac{i}{2}} = \frac{1}{\frac{2+i}{2}}$$

$$= \frac{2}{2+i} \times \frac{2-i}{2-i} = \frac{4-2i}{5} = \frac{4}{5} - \frac{2}{5}i$$

Example(20-12): Find the sum of (H.W)

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n}$$

Example(20-13): Find the sum of (H.W)

$$\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n$$

- Ratio test اختبار النسبة

Suppose $\{z_n\}$ is sequence then there exist a positive real number $L \ni$

$$L = \lim_{n\to\infty} \left|\frac{z_n + 1}{z_n}\right|$$

Then :

[1] $if\ L < 1\ then\ \sum_{n=0}^{\infty} z_n$ is converges

[2] $if\ L\ > 1\ then\ \sum_{n=0}^{\infty} z_n$ is diverges

[3] $if\ L = 1\ then$ the test is inconclusive

Example(20-14): Investigate the convergence of the following series

[1] $\sum_{n=0}^{\infty} \frac{n!}{(1+i)^n}$

Solution :

$$L = \lim_{n\to\infty} \left|\frac{z_n + 1}{z_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)!}{(1+i)^{n+1}}}{\frac{n!}{(1+i)^n}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)!}{(1+i)^{n+1}} \times \frac{(1+i)^n}{n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)n!}{(1+i)^n\,(1+i)} \times \frac{(1+i)^n}{n!}\right|$$

$$= \lim_{n\to\infty} \left|\frac{(n+1)}{(1+i)}\right| = \left|\frac{1}{1+i}\right| \lim_{n\to\infty} (n+1) = \infty$$

$$\therefore L > 1$$

$$\therefore \sum_{n=0}^{\infty} \frac{n!}{(1+i)^n} \quad is\ diverges$$

[2] $\sum_{n=0}^{\infty} \frac{2^n z^n}{n!} \quad , |z| \leq 4$

Solution :

$$L = \lim_{n\to\infty} \left|\frac{z_n+1}{z_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}z^{n+1}}{(n+1)!}}{\frac{2^n z^n}{n!}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{8^{n+1}}{(n+1)!} \times \frac{n!}{8^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{8^n 8}{(n+1)^n!} \times \frac{n!}{8^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{8}{n+1}\right| = \frac{8}{\infty}$$

$$\therefore L < 1$$

$$\therefore \sum_{n=0}^{\infty} \frac{n!}{(1+i)^n} \quad is\ converges$$

[3] $\sum_{n=0}^{\infty} \frac{i}{n^2}$

Solution :

$$L = \lim_{n\to\infty} \left|\frac{z_n+1}{z_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{i}{(n+1)^2}}{\frac{i}{n^2}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{i}{(n+1)^2} \times \frac{n^2}{i}\right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2}{n^2(1+\frac{1}{n})} \right| = 1$$

$$\therefore L = 1$$

$$\therefore \sum_{n=0}^{\infty} \frac{i}{n^2} \quad is\ inconclusiv$$

[4] $\sum_{n=0}^{\infty} \frac{(1+i)^n}{n+2}$ (H.W)

[5] $\sum_{n=1}^{\infty} \frac{1}{n^2 3^n}$ (H.W)

Complex analysis (lecture 19)

-
- **Cauchy's integral formula صيغة كوشي التكاملية**
- **Cauchy's integral formula (المشتقات النونية) صيغة كوشي التكاملية**
- **Theorem Cauchy – Courstat نظرية كوشي كورسات**

Cauchy's integral formula صيغة كوشي التكاملية

The integrals

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

لايجاد التكامل حسب كوشي

1)نجعل المقام يساوي صفر

2) ثم نجد النقاط المعزولة

3) عند ايجاد النقاط المعزولة نحدد $f(z)$ و $z-a$ وبعد ذلك نطبق القانون

Example(19-1): Evaluate the integral .

[1] $\int_C \frac{z^5}{(9-z^2)(z+i)}dz, \quad C = \{z: |z| = 2\}$ positively oriented عكس عقارب الساعة

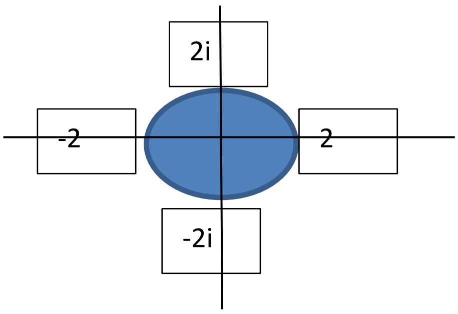
Solution :

$$9 - z^2 = 0 \Rightarrow z = \mp 3 \ outside\ circle$$

$$z + i = 0 \Rightarrow z = -i \ isolated\ point$$

$$\int_C \frac{z^5}{(9-z^2)(z+i)}dz = 2\pi i f(-i)$$

$$= 2\pi i \frac{(-i)^5}{9-(-i)^2} = 2\pi i \frac{-i}{9+1} = \frac{\pi}{5}$$



[2] $\int_C \frac{e^{z^2}}{(z-1)(z-2)} dz, \quad C = \{z: |z| = 3\}$ positively oriented عكس عقارب الساعة

Solution :

$$z - 1 = 0 \Longrightarrow z = 1 \; inside \; circle$$

$$z - 2 = 0 \Longrightarrow z = 2 \; inside \; circle$$

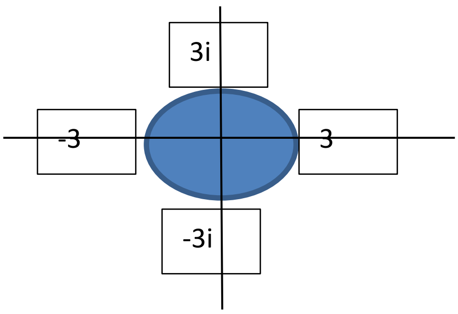
$$\int_C \frac{e^{z^2}}{(z-1)(z-2)} dz = 2\pi i f(-i)$$

$$= \int_C \left[\frac{e^{z^2}}{(z-2)} - \frac{e^{z^2}}{(z-1)} \right] dz$$

$$\int_C \frac{e^{z^2}}{(z-2)} dz - \int_C \frac{e^{z^2}}{(z-1)} dz$$

$$= 2\pi i f(2) = 2\pi i f(1)$$

$$= 2\pi i e^{2^2} - 2\pi i e^{1^2} = 2\pi i e^4 - 2\pi i e^1$$



[3] $\int_C \frac{sin3z}{z+\frac{\pi}{2}} dz, \quad C = \{z: |z| = 5\}$ in the unit circle.

Solution :

$$z+\frac{\pi}{2}=0 \Rightarrow z=-\frac{\pi}{2}$$

$$\int_C \frac{sin3z}{z+\frac{\pi}{2}}dz = 2\pi i f(-\frac{\pi}{2})$$

$$= 2\pi i \sin\left(3\left(-\frac{\pi}{2}\right)\right) = -2\pi i\ sin\left(\frac{3\pi}{2}\right)$$

$$= 2\pi i(-1) = -2\pi i$$

[4] $\int_C \frac{e^{3z}}{z-\pi i}dz, \quad C=\{z:|z|=4\}$ in the unit circle. (H.W)

Cauchy's integral formula (صيغة كوشي التكاملية (المشتقات النونية

The integrals

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a)\ , n=0,1,2,\ldots$$

<u>Example(19-2):</u> Evaluate the integral .

[1] $\int_C \frac{e^{2z}}{(z+1)^4}dz, \quad C=\{z:|z|=3\}$ in the unit circle.

Solution :

$$(z+1)^4=0 \Rightarrow z=-1\ isolated\ point$$

$$n=3$$

$$f(z)=e^{2z}$$

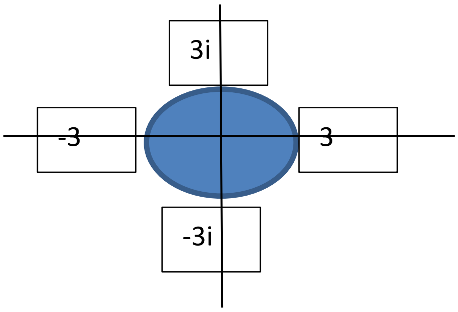
$$f^{(1)}(z)=2e^{2z}$$

$$f^{(2)}(z)=4e^{2z}$$

$$f^{(3)}(z) = 8e^{2z}$$

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f^{(3)}(-1)$$

$$= \frac{\pi i}{3}(8e^{-2}) = \frac{8\pi i}{3e^2}$$



[2] $\int_C \frac{sin^6 z}{\left(z+\frac{\pi}{6}\right)^3} dz, \quad C = \{z: |z| = 1\}$ in the unit circle.

<u>Solution :</u>

$$\left(z+\frac{\pi}{6}\right)^3 = 0 \Longrightarrow z = -\frac{\pi}{6} \ isolated\ point$$

$$n = 2$$

$$f(z) = sin^6 z$$

$$f^{(1)}(z) = 6sin^5 z\ cosz$$

$$f^{(2)}(z) = 30sin^4 z\ cos^2 z - 6sin^6 z$$

$$\int_C \frac{\sin^6 z}{\left(z+\frac{\pi}{6}\right)^3} dz = \frac{2\pi i}{2!} f^{(2)}\left(-\frac{\pi}{6}\right)$$

$$= \pi i \left(30\left(\sin\left(-\frac{\pi}{6}\right)\right)^4 \left(\cos\left(-\frac{\pi}{6}\right)\right)^2 - 6\left(\sin\left(-\frac{\pi}{6}\right)\right)^6\right)$$

$$= \pi i \left[30\left(\frac{3}{4}\right)\left(\frac{1}{16}\right) - 6\left(\frac{1}{64}\right)\right]$$

$$= \pi i \left[\frac{45}{32} - \frac{3}{32}\right] = \pi i \frac{42}{32} = \frac{21}{16}\pi i$$

[3] $\int_C \frac{e^{iz}}{(z-\pi i)^2} dz, \quad C = \{z: |z-\pi i| = 1\}$.(H.W)

Theorem Cauchy – Courstat نظرية كوشي كورسات

Let C be a simple closed contour and let $f(z)$ be a function that is analytic in the interior of C as well as on C it self the

$$\int_C f(z)dz = 0$$

[1] $\int_C f(z)dz = \int_a^b f(z)z'(t)dt$

[2] $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$

[3] $\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$, $n = 0,1,2, \dots$

Example(19-3): Show that .

[1] $\int_C \frac{z}{z+3} dz = 0, \quad in\ the\ circle\ |z| = 1$.

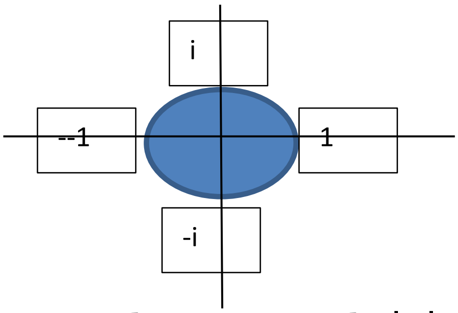
Solution :

$$z + 3 = 0 \Rightarrow z = -3$$

By Couchy Coursat

f is analytic on $C/\{-3\}$ and in the region closed by C

$$\therefore \int_C \frac{z}{z+3} dz = 0$$



[2] $\int_C e^{zi} dz = 0, \;\; if\ C\ is\ \ in\ the\ unit\ circle\ |z| = 1$.

<u>Solution :</u>

By Couchy Coursat

f is analytic on C and in the region closed by C

$$\therefore \int_C e^{zi} dz = 0$$

[3] Evaluate

$\int_C \frac{1}{z} dz\,, \;\; if\ C\ is\ the\ positively\ oriented\ circle\ |z| = 2$

(H.W)

Complex analysis (lecture 16)

-
- **Complex exponents الاسس المركبة**
- **Trigonometric functions الدوال المثلثية**

Complex exponents (z^c, c^z) الاسس المركبة

Definition (16-1): Let z be a nonzero complex number and let C be any complex number we define z^c as

$$z^c = e^{c\,logz} = e^{c\,[ln|z|+i\,(Argz+2k\pi)]} \quad ,k \in z$$

Example(16-1): Find the values of

[1] i^i

Solution :

$$i^i = e^{i\,logi} = e^{i\,[ln|i|+i\,(Argi+2k\pi)]}$$

$$= e^{i\left[o+i\left(tan^{-1}\left(\frac{1}{0}\right)+2k\pi\right)\right]}$$

$$= e^{i\left[i\left(\frac{\pi}{2}+2k\pi\right)\right]}$$

$$= e^{-\left(\frac{\pi}{2}+2k\pi\right)} \quad ,k \in z$$

[2]$(2i)^i$

Solution :

$$e^{i\,\log(2i)} = e^{i\,[ln|2i|+i\,(Arg(2i)+2k\pi)]}$$

$$= e^{i\left[ln2+i\left(tan^{-1}\left(\frac{2}{0}\right)+2k\pi\right)\right]}$$

$$= e^{i\left[in2+i\left(\frac{\pi}{2}+2k\pi\right)\right]}$$

$$= e^{i\,ln2-\left(\frac{\pi}{2}+2k\pi\right)}$$

$$= \frac{2^i}{e^{\frac{\pi}{2}+2k\pi}} \quad ,k \in z$$

[3] $(-3i)^{-i}$ (H.W)

[4] $(-i)^{2i}$ (H.W)

[5] $(-1)^{\frac{2}{3}}$

Solution :

$$e^{\frac{2}{3}\log(-1)} = e^{\frac{2}{3}[ln|-1|+i\,(Arg(-1)+2k\pi)]}$$

$$= e^{\frac{2}{3}\left[\,i\left(\,tan^{-1}\left(\frac{0}{-1}\right)+2k\pi\right)\right]}$$

$$= e^{\frac{2}{3}[\,i\,(\,\pi+2k\pi)]}$$

$$= e^{\frac{(\,2\pi+4k\pi)}{3}i}$$

$$k = 0,1,2$$

$$k = 0 \Longrightarrow e^{\frac{2\pi i}{3}} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$$k = 1 \Longrightarrow e^{2\pi i} = 1$$

$$k = 2 \Longrightarrow e^{\frac{10\pi i}{3}} = e^{\frac{4\pi i}{3}} = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

[6] $(1-\sqrt{3}i)^{\frac{3}{2}}$ (H.W)

Definition (16-2) The principal value of z^c is

$$z^c = e^{c\,logz}$$

Example(16-2): Find the principal value of $(1+i)^{2+i}$

Solution :

$$(1+i)^{2+i} = e^{(2+i)\log(1+i)}$$

$$= e^{(2+i)\,[ln|1+i|+i\,(Arg(1+i))]}$$

$$= e^{(2+i)\left[ln\sqrt{2}+i\,\frac{\pi}{4}\right]}$$

$$= e^{ln2+\frac{\pi}{2}i+i\,ln\sqrt{2}-\frac{\pi}{4}}$$

$$= e^{ln2}.e^{\frac{\pi}{2}}.e^{ln\sqrt{2}^{i}} - e^{-\frac{\pi}{4}}$$

$$= \frac{2i\,\sqrt{2}^{i}}{e^{\frac{\pi}{4}}}$$

$$c^{z} = e^{zlogc} \qquad c \neq 0$$

Example(16-3): Find the values of

[1] 2^{1+i}

Solution :

$$= e^{(1+)i\,log2} = e^{(1+i)\,[ln|2|+i\,(Arg2+2k\pi)]}$$

$$= e^{(1+i)\,[ln2+i\,(0+2k\pi)]}$$

$$= e^{(1+i)\,[ln2+2k\pi i]}$$

$$= e^{ln2+2k\pi i+i\,ln2-\,2k\pi}$$

$$= e^{ln2}.e^{2k\pi i}.e^{ln2^{i}} - e^{-2k\pi}$$

$$= \frac{2^{i+1}}{e^{2k\pi}}$$

[2] $(\,\pi)^{i}$ (H.W)

Trigonometric functions الدوال المثلثية

$$sinz = \frac{e^{iz} - e^{-iz}}{2i}\ , \quad cosz = \frac{e^{iz} + e^{-iz}}{2}$$

$$tanz = \frac{sinz}{cosz} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})},$$

<u>Properties of sin and cos for complex number</u>

[1] $\sin(z + 2k\pi) = sinz$

Proof:

L.H.S

$$\sin(z + 2k\pi) = \frac{e^{i(z+2k\pi)} - e^{-i(z+2k\pi)}}{2i}$$

$$= \frac{e^{iz+2k\pi i} - e^{-iz-2k\pi i}}{2i}$$

$$= \frac{e^{iz}.e^{2k\pi i} - e^{-iz}.e^{-2k\pi i}}{2i}$$

$$= \frac{e^{iz} - e^{-iz}}{2i} = sinz \quad R.H.S$$

[2] $\sin(z + \pi) = -sinz$

[3] $\sin\left(z + \frac{\pi}{2}\right) = cosz$

[4] $\frac{d}{dz}(sinz) = cosz$

Proof:

$$\frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{i(e^{iz} + e^{-iz})}{2i}$$

$$= \frac{e^{iz} + e^{-iz}}{2} = cosz$$

[5] $sin(z_1 + z_2) = sinz_1cosz_2 + cosz_1sinz_2$

Proof:

R.H.S

$$sinz_1cosz_2 + cosz_1sinz_2$$

$$= \left(\frac{e^{i\,z_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{i\,z_2} + e^{-iz_2}}{2}\right) + \left(\frac{e^{i\,z_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{i\,z_2} - e^{-iz_2}}{2i}\right)$$

$$= \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(-z_1+z_2)} - e^{-i(z_1+z_2)}}{4i} + \frac{e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{i(-z_1+z_2)} - e^{-i(z_1+z_2)}}{4i}$$

$$= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)} + e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{4i}$$

$$= \frac{2\left[e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}\right]}{4i} = sin(z_1 + z_2) \quad L.H.S$$

[6] $sin(z_1 - z_2) = sinz_1cosz_2 - cosz_1sinz_2$ (H.W)

[7] $cos(z_1 + z_2) = cosz_1cosz_2 - sinz_1sinz_2$

Proof:

R.H.S

$$cosz_1cosz_2 - sinz_1sinz_2 = \left(\frac{e^{i\,z_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{i\,z_2} + e^{-iz_2}}{2}\right) - \left(\frac{e^{i\,z_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{i\,z_2} - e^{-iz_2}}{2i}\right)$$

$$= \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{i(-z_1+z_2)} + e^{-i(z_1+z_2)}}{4}$$

$$+ \frac{e^{i(z_1+z_2)} - e^{i(z_1-z_2)} - e^{i(-z_1+z_2)} + e^{-i(z_1+z_2)}}{4}$$

$$= \frac{2\left[e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}\right]}{4i} = cos(z_1+z_2) \quad L.H.S$$

[8] $cos(z_1 - z_2) = cosz_1cosz_2 + sinz_1sinz_2$ (H.W)

[9] $\frac{d}{dz}(cosz) = -sinz$

Proof:

$$\frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{ie^{iz} - ie^{-iz}}{2} = \frac{i(e^{iz}-e^{-iz})}{2} \cdot \frac{i}{i}$$

$$= \frac{i^2(e^{iz}-e^{-iz})}{2i} = -\frac{e^{iz}-e^{-iz}}{2i} = -sinz$$

[10] $\cos(-z) = cosz$

[11] $\cos(z + 2k\pi) = cosz$

[12] $\cos(z+\pi) = -cosz$

[13] $sis2z = 2sinzcosz$

[14] $cos2z = cos^2z - sin^2z$

[15] $\overline{cosz} = cos\bar{z}$

Proof:

$$\overline{\left(\frac{e^{iz}+e^{-iz}}{2}\right)} = \frac{\overline{e^{iz}+e^{-iz}}}{\bar{2}} = \frac{e^{i\bar{z}} + e^{\overline{-iz}}}{2}$$

$$= \frac{e^{i\bar{z}} + e^{-i\bar{z}}}{2} = cos\bar{z}$$

[16] $\overline{\cos(zi)} = \cos(\bar{z}i)$

[17] $\overline{\sin(z)} = \sin(\bar{z})$

[18] $\overline{\sin(zi)} = \sin(\bar{z}i)$

Example(16-4): prove that

[1] $1 + tan^2 z = sec^2 z$

Solution : L.H.S

$$1 + tan^2 z = 1 + \frac{sin^2 z}{cos^2 z}$$

$$= 1 + \frac{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2}{\left(\frac{e^{iz} + e^{-iz}}{2}\right)^2} = 1 + \frac{\frac{\left(e^{iz} - e^{-iz}\right)^2}{-4}}{\frac{(e^{iz} + e^{-iz})^2}{4}}$$

$$= \frac{\left(e^{iz} + e^{-iz}\right)^2 - \left(e^{iz} - e^{-iz}\right)^2}{e^{2iz} + 2 + e^{-2iz}}$$

$$= \frac{e^{2iz} + 2 + e^{-2iz} - \left[e^{2iz} - 2 + e^{-2iz}\right]}{e^{2iz} + 2 + e^{-2iz}}$$

$$= \frac{e^{2iz} + 2 + e^{-2iz} - e^{2iz} + 2 - e^{-2iz}}{e^{2iz} + 2 + e^{-2iz}}$$

$$\frac{4}{(e^{iz} + e^{-iz})^2} = \left(\frac{2}{e^{iz} + e^{-iz}}\right)^2 = sec^2 z$$

[2] $1 + cot^2 z = csc^2 z$ (H.W)

Example(16-5): Evaluate

[1] $\sin(i)$

Solution :

$$\sin(i) = \frac{e^{i(i)} - e^{-i(i)}}{2i}$$

$$= \frac{e^{i^2} - e^{-i^2}}{2i} = \frac{e^{-1} - e^{1}}{2i} = \frac{\frac{1}{e} - e^{1}}{2i}$$

$$\frac{\frac{1-e^2}{e}}{2i} = \frac{1-e^2}{2ei}$$

[2] $\tan(2i)$ (H.W)

Example(16-6): Solve the equations

[1] $\cos z = 4$

Solution :

$$\frac{e^{iz} + e^{-iz}}{2} = 4$$

$$= e^{iz} + \frac{1}{e^{iz}} = 8 \Big] * e^{iz}$$

$$e^{2iz} + 1 = 8e^{iz} \Rightarrow e^{2iz} - 8e^{iz} + 1 = 0$$

$$x = \frac{-b\sqrt{b^2 - 4ac}}{2a} \Rightarrow e^{iz} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$e^{iz} = \frac{8 \mp \sqrt{64 - 4}}{2} = \frac{8 \mp \sqrt{60}}{2}$$

$$e^{iz} = 4 \mp \sqrt{15} \Rightarrow lne^{iz} = \ln(4 \mp \sqrt{15})$$

$$\Rightarrow iz = \left(4 \mp \sqrt{15}\right)$$

$$z = \frac{\left(4 \mp \sqrt{15}\right)}{i} * \frac{-i}{-i} \Rightarrow z$$

$$= -i\ln\left(4 \mp \sqrt{15}\right) + 2k\pi \, , k = 0, \mp 1, \ldots$$

[2] $\sin z = \frac{1}{2i}$ (H.W)

Complex analysis (lecture 18)

-
- **Complex integration التكامل المركب**
- **Contour integral التكامل الخطي**
- **Integral inequality (ML- inequality) تكامل عدم المساواة**

Complex integration التكامل المركب

The integrals

$$\int_a^b z(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Example(18-1): let $f(t) = (1+it)^2$, $t \in [0,1]$ find the integral .

Solution :

$$f(t) = 1 + 2it - t^2 = 1 - t^2 + 2it$$

$$\int_0^1 f(t)dt = \int_0^1 (1-t^2)dt + i\int_0^1 2tdt$$

$$= t - \frac{t^3}{3}\Big]_0^1 + it^2\Big]_0^1 = \left(1-\frac{1}{3}\right) + i(1) = \frac{2}{3} + i$$

Example(18-2): let $f(t) = e^{it}$, $0 \le t \le \frac{\pi}{4}$ find the integral .

Solution :

$$\int_0^{\frac{\pi}{4}} e^{it}\,dt = \frac{1}{i}e^{it}\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{i}\cdot\frac{-i}{-i}\left[e^{i\frac{\pi}{4}} - e^0\right] = e^{-1}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} - 1\right]$$

$$= -\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}} + i = \frac{1}{\sqrt{2}} + \left(1 - \frac{1}{\sqrt{2}}\right)i$$

Example(18-3): Evaluate the integrals $\int_0^{\frac{\pi}{6}} e^{2it}dt$ (H.W)

Example(18-4): Evaluate the integrals and then identifying the real and imaginary parts of the value found .

$$\int_0^\pi e^{(1+i)t}dt$$

Solution :

$$\int_0^\pi e^{(1+i)t}dt = \int_0^\pi e^t . e^{it}dt$$

$$= \int_0^\pi e^t(cost + isint)dt = \int_0^\pi (cost\ e^t + isint\ e^t)dt$$

$$= \int_0^\pi (cost\ e^t\ )dt + i\int_0^\pi (\ sint\ e^t)dt$$

$$I_1 = \int_0^\pi (cost\ e^t\ )dt$$

$$u = e^t \ \Longrightarrow du = e^t dt$$

$$dv = cost \Longrightarrow v = sint$$

$$I_1 = uv - \int vdu$$

$$= e^t\ sint\ ]_0^\pi - \int_0^\pi (\ sint\ e^t)dt$$

$$= (e^\pi sin\pi - e^0 sin0\ ) - \left[e^t(-cost)]_0^\pi + cost\ e^t dt\right]$$

$$u = e^t \ \Longrightarrow du = e^t dt$$

$$dv = sint \Longrightarrow v = -cost$$

$$I_1 = uv - \int vdu$$

$$= e^t \, cost \,]_0^{\pi} - \int_0^{\pi} (\, cost \, e^t) dt$$

$$I_1 + I_2 = (e^{\pi} cos\pi - e^0 cos0 \,)$$

$$2I_1 = (-e^{\pi} - 1)] \div 2$$

$$I_1 = \frac{-(\, e^{\pi} + 1)}{2} = -\frac{1}{2}(\, e^{\pi} + 1) \, real \, part$$

$$I_2 = \int_0^{\pi} (sint \, e^t \,) dt$$

$$u = e^t \Longrightarrow du = e^t dt$$

$$dv = sint \Longrightarrow v = -cost$$

$$I_2 = uv - \int vdu$$

$$= -e^t \, cost \,]_0^{\pi} - \int_0^{\pi} (cost \, e^t) dt$$

$$I_2 = (-e^{\pi} cos\pi + e^0 cos0 \,) + e^t (sint)]_0^{\pi} - \int_0^{\pi} (sint \, e^t \,) dt$$

$$u = e^t \Longrightarrow du = e^t dt$$

$$dv = cost \Longrightarrow v = sint$$

$$I_1 + I_2 = (e^{\pi} + 1 \,) + 0$$

$$2I_2 = (\, e^{\pi} + 1)] \div 2$$

$$I_2 = \frac{(\, e^{\pi} + 1)}{2}$$

$$\therefore Re \int_0^{\pi} e^{(1+i)t} dt = -\frac{1}{2}(\, e^{\pi} + 1)$$

$$Im \int_0^{\pi} e^{(1+i)t} dt = \frac{1}{2}(\, e^{\pi} + 1)$$

Contour integrals (line integral) التكامل الخطي

Let $f(z) = u + iv$ continuous in all point of the curve C then

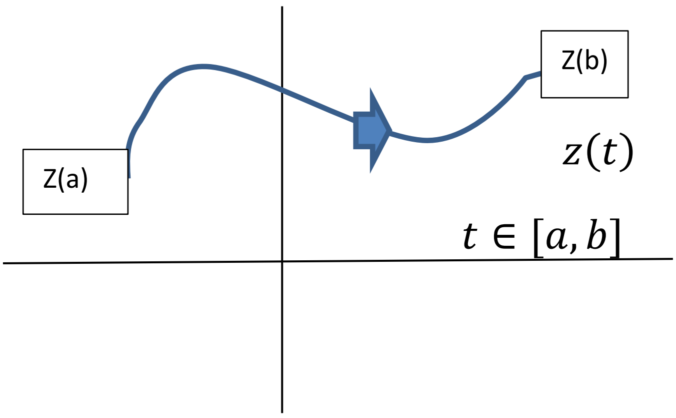
$$\int_a^b f(z)dz = \int_a^b f(z(t)).z'(t)dt$$

Properties of line integral

[1] $\int_{-c} f(z)dz = -\int_c f(z)dz$

[2] $\int_c z_0 f(z)dz = z_0 \int_c f(z)dz$

[3] $\int_c [f(z) \mp g(z)]\, dz = \int_c f(z)dz \mp \int_c g(z)dz$



Example(18-5): find out the value of

[1] $\frac{dz}{z-a}$, $z(t) = a + re^{it}$, $0 \le t \le 2\pi$

Solution :

$$z'(t) = rie^{it}dt$$

$$\int_c f(z)dz = \int_a^b f(z(t)).z'(t)dt$$

$$= \int_0^{2\pi} \frac{1}{a + re^{it} - a}.ri\, e^{it}dt$$

$$= \int_0^{2\pi} \frac{rie^{it}}{re^{it}}\, dt = \int_0^{2\pi} i\, dt$$

$$= i\int_0^{2\pi} dt = it\Big]_0^{2\pi} = i[2\pi - 0] = 2i\pi$$

[2] $\frac{dz}{z-a}$, $z(t) = are^{-it}$, $-2\pi \le t \le 0$

Solution :

$$z'(t) = -rie^{-it}dt$$

$$\int_c f(z)dz = \int_a^b f(z(t)).z'(t)dt$$

$$= \int_{-2\pi}^{0} \frac{1}{a + re^{-it} - a}.(-ri\, e^{it})dt$$

$$= -\int_{0}^{2\pi} -i\, dt = 2i\pi$$

[3] $\int_c \frac{z+2}{z} dz, c: [\, z = 2e^{i\theta}, \quad 0 \le \theta \le \pi]$ (H.W)

[4] $\int_c \bar{z}\, dz, z(\theta) = 2e^{i\theta} \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Solution :

$$z'(t) = 2ie^{i\theta} d\theta$$

$$\int_c f(z)dz = \int_a^b f(z(t)).z'(t)dt$$

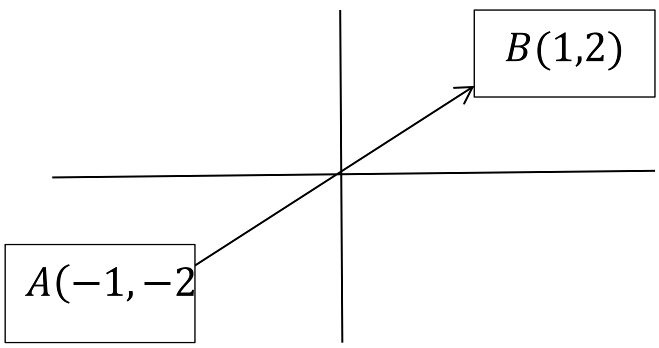
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overline{2e^{\iota\theta}}.(2ie^{i\theta})d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\ \overline{e^{\iota\theta}}.(\,ie^{i\theta})d\theta$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\, i.e^{-i\theta}e^{i\theta}d\theta = 4i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = 4i[\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4i\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 4i\pi$$

Example(18-6): $\int_c e^z dz$ where C is the curve $y = 2x$ from $A(-1,-2), to\ B(1,2)$

Solution :



$$y = 2x \quad let\ x = t\ , y = 2t$$

$$z(t) = x + yi$$

$$z(t) = t + 2ti \quad , t \in [-1,1]$$

$$z'(t) = 1 + 2i$$

$$\int_c f(z)dz = \int_a^b f(z(t)).z'(t)dt$$

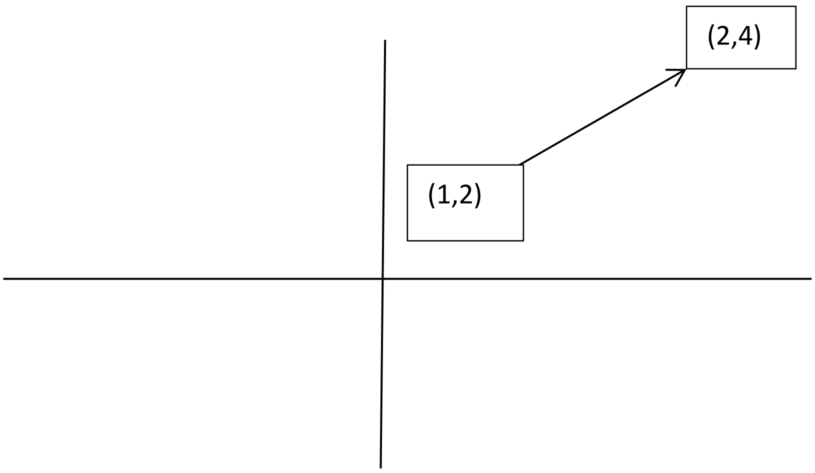
$$= \int_{-1}^{1} e^{1+2ti}.(1+2i)dt$$

$$= e^{1+2ti}\Big]_{-1}^{1} = e^{1+2ti} - e^{-1-2i}$$

Example(18-7): $\int_c \bar{z}^2 dz$ where C is the curve $y = x^2$ from $A(0,0), to\ B(1,1)$ (H.W)

Example(18-8): Evaluate $\int_c \frac{1}{z} dz$ where C is the line segment from $(1,2), to\ B(2,4)$

Solution :



$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-2}{x-1}=\frac{4-2}{2-1}$$

$$y-2=2x-2 \implies y=2x$$

$$let\ y=t\ ,x=\frac{1}{2}t$$

$$z(t)=x+yi$$

$$z(t)=\frac{1}{2}t+ti$$

$$z'(t)=\frac{1}{2}+i\ \ ,t\in[2,4]$$

$$\int_c f(z)dz=\int_a^b f(z(t)).z'(t)dt$$

$$\int_c f(z)dz=\int_2^4 \frac{1}{\frac{1}{2}t+ti}.\left(\frac{1}{2}+i\right)dt$$

$$=ln\left|\frac{1}{2}t+ti\right|]\ \begin{matrix}4\\2\end{matrix}=ln|2+4i|-ln|1+2i|$$

$$=ln\left|\frac{2+4i}{1+2i}\right|$$

Integral inequality (ML- inequality) تكامل عم المساواة

$$\left|\int_c f(z)dz\right|\leq ML$$

$$L=\int_a^b |f'(t)|dt \quad\ or\ \ L=r\theta$$

$$|f(z)| \leq M$$

قوانين مهمه

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$|z_1 - z_2| \leq |z_1| - |z_2|$ تستخدم في البسط

$$|z_1 + z_2| \geq |z_1| + |z_2|$$

$|z_1 - z_2| \geq |z_1| - |z_2|$ تستخدم في المقام

Example(18-9): Suppose C is contour

i.e $z(\theta) = 3e^{i\theta}$ $0 \leq \theta \leq \pi$

find out the upper bound of integral $\int_c \frac{z^{\frac{1}{2}}}{z^2+1} dz$

Solution :

$$L = \int_a^b |z'(\theta)| d\theta \quad , z'(\theta) = 3ie^{i\theta} \quad , |z| = 3$$

$$L = \int_0^\pi \left|3ie^{i\theta}\right| d\theta = \int_0^\pi 3\, d\theta = 3\theta\Big]_0^\pi = 3\pi$$

$$|f(z)| \leq M$$

$$\left|\frac{z^{\frac{1}{2}}}{z^2+1}\right| \leq \frac{\left|z^{\frac{1}{2}}\right|}{|z^2+1|} \leq \frac{\sqrt{3}}{9-1} \leq \frac{\sqrt{3}}{8}$$

$$\therefore \boldsymbol{M} = \frac{\sqrt{3}}{8} \quad , L = 3\pi$$

$$\therefore \left|\int_c \frac{z^{\frac{1}{2}}}{z^2+1} dz\right| \leq ML \leq 3\pi.\frac{\sqrt{3}}{8} \leq 3\frac{\sqrt{3}}{8}\pi$$

Example(18-10): find out the upper estimate the integral

$$\int_c \frac{z^2+3}{z^2+2}\, e^{iz} dz$$

Where $z(\theta) = 2e^{i\theta}$ $0 \leq \theta \leq \frac{\pi}{3}$

Solution :

$$L = r\theta = \frac{2\pi}{3}$$

$$|f(z)| \leq M$$

$$\left|\frac{z^2+3}{z^2+2}\, e^{iz}\right| \leq \frac{|z^2+3|}{|z^2+2|}\, |e^{iz}|$$

$$\leq \frac{|z^2+3|}{|z^2+2|} \leq \frac{|z^2|+3}{|z^2|-2} \leq \frac{4+3}{4-2} \leq \frac{7}{2}$$

$$M = \frac{7}{2}\ ,\ \ L = \frac{2\pi}{3}$$

$$\left|\int_c \frac{z^2+3}{z^2+2}\, e^{iz} dz\right| \leq ML \leq \frac{7}{2}.\frac{2\pi}{3} \leq \frac{7\pi}{3}$$

Example(18-11): find out the upper estimate the integral

$$\int_c \frac{(z-1)}{z^2-7}\, e^{2z} dz$$

Where $z(\theta) = e^{i\theta}$ $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$ (H.W)

Complex analysis (lecture 17)

-
- **Inverse trigonometric functions معكوس الدوال المثلثية**
- **Derivative of inverse trigonometric functions مشتقة معكوس الدوال المثلثية**
- **Hyperbolic trigonometric functions الدوال المثلثية الزائدية**
- **Derivative of hyperbolic trigonometric functions مشتقة الدوال المثلثية الزائدية**
- **Inverse hyperbolic trigonometric functions معكوس الدوال المثلثية الزائدية**

Inverse trigonometric functions

[1] $sin^{-1}z = -i\ log\left[iz + \sqrt{1-z^2}\right]$

[2] $cos^{-1}z = -i\ log\left[z + \sqrt{z^2-1}\right]$

[3] $tan^{-1}z = \frac{1}{2i} log\left[\frac{1+iz}{1-iz}\right]$

[4] $cot^{-1}z = \frac{1}{2i} log\left[\frac{z+i}{z-i}\right]$

[5] $sec^{-1}z = -i\ log\left[\frac{1+\sqrt{1-z^2}}{z}\right]$ (H.W)

[6] $csc^{-1}z = -i\ log\left[\frac{i+\sqrt{z^2-1}}{z}\right]$

Proof [1] :

$$sin^{-1}z = -i\ log\left[iz + \sqrt{1-z^2}\right]$$

$$w = sin^{-1}z \implies sinw = z$$

$$\frac{e^{iw} - e^{-iw}}{2i} = z \implies e^{iw} - \frac{1}{e^{iw}} = 2iz\Big] * e^{iw}$$

$$e^{2iw} - 2ize^{iw} - 1 = 0$$

$$e^{iw} = \frac{-b \mp \sqrt{b^2-4ac}}{2a} = \frac{2iz + \sqrt{(-2iz)^2 - 4(1)(-1)}}{2(1)}$$

$$e^{iw} = \frac{2iz + \sqrt{-4z^2+4}}{2}$$

$$e^{iw} = \mathrm{iz} + \sqrt{1-z^2}$$

$$loge^{iw} = \log\left[\mathrm{iz} + \sqrt{1-z^2}\right]$$

$$iw = \log\left[\mathrm{iz} + \sqrt{1-z^2}\right]$$

$$w = \frac{1}{i}\log\left[\mathrm{iz} + \sqrt{1 - z^2}\right] \qquad \Big] * \frac{-i}{-i}$$

$$w = -i\ log\left[\mathrm{iz} + \sqrt{1 - z^2}\right]$$

Proof [3] :

$tan^{-1}z = \frac{1}{2i} log\left[\frac{1+iz}{1-iz}\right]$

$$w = tan^{-1}z \ \Rightarrow tanw = z$$

$$\frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} = z \ \Rightarrow \ e^{iw} - \frac{1}{e^{iw}} = \ iz\left(e^{iw} + \frac{1}{e^{iw}}\right)\Big] * \ e^{iw}$$

$$e^{2iw} - \ 1 = \ ize^{iw} + zi$$

$$e^{2iw} - \ ize^{iw} \ = \ 1 + zi$$

$$e^{2iw}\ (1 - zi) \ = \ 1 + zi$$

$$e^{2iw} \ = \ \frac{1 + zi}{1 - zi}$$

$$2iw = \log\left[\frac{1 + zi}{1 - zi}\right]$$

$$w = \frac{1}{2i}\log\left[\frac{1 + zi}{1 - zi}\right]$$

Example(17-1): Evaluate the following

[1] $cos^{-1}\sqrt{3}$

Solution :

$$cos^{-1}z = -i\ log\left[z + \sqrt{\ z^2 - 1}\right]$$

$$cos^{-1}\sqrt{3} = -i\ log\left[\sqrt{3} + \sqrt{\ 2}\right]$$

$$= -i\ log\left(\left[\sqrt{3} + \sqrt{\ 2}\right] + 2k\pi i\right) \quad k\ = 0, \mp 1, \ldots$$

[2] $sin^{-1}(1+\sqrt{3}i)$

Solution :

$$sin^{-1}z = -i\ log\left[iz + \sqrt{1-z^2}\right]$$

$$sin^{-1}(1+\sqrt{3}i) = -i\ log\left[i(1+\sqrt{3}i) + \sqrt{1-(1+\sqrt{3}i)^2}\right]$$

$$= -i\ log\left[i-\sqrt{3} + \sqrt{1-(1+2\sqrt{3}i+(-\sqrt{3}))}\right]$$

$$= -i\ log\left[i-\sqrt{3} + \sqrt{1-1-2\sqrt{3}i+\sqrt{3}}\right]$$

$$= -i\ log\left[i-\sqrt{3} + \sqrt{\sqrt{3}-2\sqrt{3}i}\right]$$

Derivative of inverse trigonometric functions

[1] $\frac{d}{dz}(sin^{-1}z) = \frac{1}{\sqrt{1-z^2}}$

[2] $\frac{d}{dz}(cos^{-1}z) = \frac{-1}{\sqrt{1-z^2}}$

[3] $\frac{d}{dz}(tan^{-1}z) = \frac{1}{1+z^2}$

[4] $\frac{d}{dz}(cot^{-1}z) = \frac{-1}{1+z^2}$

[5] $\frac{d}{dz}(sec^{-1}z) = \frac{1}{z\sqrt{z^2-1}}$

[6] $\frac{d}{dz}(csc^{-1}z) = \frac{-1}{z\sqrt{z^2-1}}$ (H.W)

Proof [1] : $\frac{d}{dz}(sin^{-1}z) = \frac{1}{\sqrt{1-z^2}}$

$$\frac{d}{dz}\left(-i\ log\left[iz+\sqrt{1-z^2}\right]\right)$$

$$= -i\frac{\left(i + \frac{-2z}{2\sqrt{1-z^2}}\right)}{iz + \sqrt{1-z^2}}$$

$$= \frac{1 - \frac{zi}{\sqrt{1-z^2}}}{iz + \sqrt{1-z^2}}$$

$$= \frac{\sqrt{1-z^2} + zi}{\sqrt{1-z^2}\left(iz + \sqrt{1-z^2}\right)} = \frac{1}{\sqrt{1-z^2}}$$

Proof [3] : $\frac{d}{dz}(tan^{-1}z) = \frac{1}{1+z^2}$

$$\frac{d}{dz}\left(\frac{1}{2i} log\left[\frac{1+iz}{1-iz}\right]\right)$$

$$= \frac{1}{2i}\frac{\left(\frac{(1-iz)i-(1+iz)(-i)}{(1-iz)^2}\right)}{\frac{1+iz}{1-iz}}$$

$$= \frac{1}{2i}\frac{i+z+i-z}{(1+iz)(1-iz)}$$

$$= \frac{2i}{2i(\ 1+z^2)} = \frac{1}{1+z^2}$$

Hyperbolic trigonometric functions

$sinhz = \frac{e^z-e^{-z}}{2}$ $, coshz = \frac{e^z+e^{-z}}{2}$

[1] $cosh^2z - sinh^2z = 1$

Proof:

$$cosh^2z - sinh^2z$$

$$= \left(\frac{e^z+e^{-z}}{2}\right)^2 - \left(\frac{e^z-e^{-z}}{2}\right)^2$$

$$= \frac{e^{2z}+2+e^{-2z}}{4} - \frac{e^{2z}-2+e^{-2z}}{4}$$

$$= \frac{e^{2z}+2+e^{-2z}-e^{2z}+2-e^{-2z}}{4} = \frac{4}{4} = 1$$

[2] $\cosh(-z) = coshz$

[3] $\sinh(-z) = -sinhz$

[4] $\sin(iz) = isinhz$

[5] $\cos(iz) = coshz$

[6] $coshz = cosy\ sinhx + isiny\ sinhx$

[7] $sinhz = cosy\ sinhx + isiny\ coshx$

[8] $|cosz|^2 = cos^2x + sinh^2y$

[9] $|\,sin^2z| = sin^2x + sinh^2y$

[10] $|sinhz|^2 = sinh^2x + sin^2y$

[11] $|coshz|^2 = sinh^2x + cos^2y$

[12] $\sinh(z_1+z_2) = sinhz_1coshz_2 + sinhz_2coshz_1$

[13] $\cosh(z_1+z_2) = coshz_1coshz_2 + sinhz_1sinhz_2$

[14] $\cosh(z+2\pi i) = coshz$

[15] $\sinh(z+2\pi i) = sinhz$

<u>**Example(17-2):**</u> Proof that

[1] $sinh^2\,\frac{z}{2} = \frac{1}{2}[coshz-1]$

<u>**Solution :**</u>

$$sinh^2\,\frac{z}{2} = \left[\frac{e^{\frac{z}{2}} - z^{-\frac{z}{2}}}{2}\right]^2 = \frac{\left(e^{\frac{z}{2}}\right)^2 - 2 + \left(e^{-\frac{z}{2}}\right)^2}{4}$$

$$= \frac{e^z - 2 + e^{-z}}{4} = \frac{1}{2}[coshz - 1]$$

[1] $cosh^2\,\frac{z}{2} = \frac{1}{2}[coshz + 1]$ (H.W)

Example(17-3): Evaluate

[1] $sinh\,\frac{\pi}{2}i$

Solution : $sinhz = \frac{e^z - e^{-z}}{2}$

$$sinh\,\frac{\pi}{2}i = \frac{e^{\frac{\pi}{2}i} - e^{-\frac{\pi}{2}i}}{2} = \frac{2i}{2} = i$$

[2] $\sinh(1 + \pi i)$ (H.W)

Example(17-4): Solve the equations :

[1] $coshz = \frac{1}{2}$

Solution : $coshz = \frac{e^z + e^{-z}}{2}$

$$\frac{e^z + e^{-z}}{2} = \frac{1}{2} \implies e^z + \frac{1}{e^z} = 1$$

$$e^{2z} - e^z + 1 = 0 \quad , z = x + yi$$

$$e^z = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{1 \mp \sqrt{1 - 4}}{2}$$

$$e^z = \frac{1 \mp \sqrt{-3}}{2} = \frac{1 \mp \sqrt{3}i}{2}$$

$$|e^z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{1} = 1$$

$$e^x = 1 \Longrightarrow lne^x = ln1 \Longrightarrow x = 0$$

$$Arg(e^z) = Imz + 2k\pi$$

$$Arg\left(\frac{1}{2} \mp \frac{\sqrt{3}}{2}\right) = y \Longrightarrow \frac{\pi}{3} = y$$

$$\therefore y = \frac{\pi}{3} + 2k\pi$$

$$z = x + yi = \left(\frac{\pi}{3} + 2k\pi\right) i$$

[2] $tanhz = \frac{1}{2}$ (H.W)

Derivative of hyperbolic trigonometric functions

[1] $\frac{d}{dz}(sinhz) = cosh$

[2] $\frac{d}{dz}(coshz) = sinhz$

[3] $\frac{d}{dz}(tanhz) = sech^2 z$

[4] $\frac{d}{dz}(cothz) = -csch^2 z$

[5] $\frac{d}{dz}(sechz) = -sechz\ tanhz$

[6] $\frac{d}{dz}(cschz) = -cschz\ cothz$

Inverse hyperbolic trigonometric functions

[1] $sinh^{-1}z = log\left[z + \sqrt{z^2 + 1}\right]$

[2] $cosh^{-1}z = log\left[z + \sqrt{z^2 - 1}\right]$

[3] $tanh^{-1}z = \frac{1}{2} log\left[\frac{1+z}{1-z}\right]$

[4] $coth^{-1}z = \frac{1}{2} log\left[\frac{z+i}{z-i}\right]$

[5] $sech^{-1}z = \; log\left[\frac{1+\sqrt{z^2-1}}{z}\right]$

[6] $csch^{-1}z = \; log\left[\frac{1+\sqrt{1+z^2}}{z}\right]$

<u>Example(17-5):</u> Find the value of the following

[1] $sinh^{-1}i$

<u>Solution :</u>

$$sinh^{-1}z = \; log\left[z+\sqrt{z^2+1}\right]$$

$$sinh^{-1}i = log\left[i+\sqrt{i^2+1}\right]$$

$$= log\left[i+\sqrt{-1+1}\right] = logi$$

$$\therefore logi = \log(1) + \left(\frac{\pi}{2}+2k\pi\right)i = \left(\frac{\pi}{2}+2k\pi\right)i$$

[2] $cosh^{-1}\frac{\pi}{4}i$ (H.W)

Complex analysis (lecture 15)

-
- **The exponential function** الدالة الاسية
- **The logarithm** الدالة اللوغارتمية

- - The exponential function الدالة الاسية

$$e^z = \exp(z) = e^{x+yi} = e^x . e^{yi} = e^x[cosy + isiny]$$

Example(15-1): Express in the form $x + yi$

[1] $e^{1+\frac{\pi}{6}i}$

Solution :

$$e^1 . e^{\frac{\pi}{6}i}$$

$$= e\left[cos\frac{\pi}{6} + isin\frac{\pi}{6}\right] = e\left[\frac{\sqrt{3}}{2} + i\frac{1}{2}\right]$$

$$= e\frac{\sqrt{3}}{2} + i\frac{e}{2}$$

[2] $e^{2+3\pi i}$

Solution :

$$e^2 . e^{3\pi i}$$

$$= e^2[cos3\pi + isin3\pi]$$

$$= e^2[\cos(3\pi - 2\pi) + isin(3\pi - 2\pi)\ ]$$

$$= e^2(-1 + 0) = e^2$$

[3] $e^{z+\pi i}$

Solution :

$$e^z . e^{\pi i}$$

$$= e^z[cos\pi + isin\pi]$$

$$= -e^z$$

[4] $e^{\sqrt{3}-\frac{\pi}{4}i}$ (H.W)

Proposition(15-1): Let $z = x + yi$ then

[1] $e^{z_1+z_2} = e^{z_1}.e^{z_2}$

[2] $e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}$

[3] $|e^z| = e^x$

[4] $\arg(e^z) = Imz + 2k\pi \quad , \ k \in z$

$$= y + 2k\pi$$

[5] $(e^z)^n = e^{nz} \quad ,n \in z$

[6] $e^z \neq 0 \quad \forall z \in \mathbb{C}$

[7] $\frac{d}{dx}[e^z] = e^z$

[8] e^z is analytic function an entire function

Proof: [3]

$$|e^{x+yi}| = |e^x.e^{yi}| = |e^x|.|e^{yi}|$$

$$= e^x|cosy + isiny| = e^x\sqrt{cos^2y + sin^2y} = e^x\sqrt{1} = e^x$$

Proof: [1]

$e^{z_1+z_2} = e^{z_1}.e^{z_2}$

$$lne^{z_1+z_2} = ln(e^{z_1}.e^{z_2})$$

$$z_1 + z_2 = lne^{z_1} + lne^{z_2}$$

$$z_1 + z_2 = z_1 + z_2$$

Example(15-2): prove that $e^{\bar{z}} = \overline{e^z}$ let $z = x + yi$

Proof:

$$\overline{(e^z)} = \overline{(e^{x+yi})} = \overline{(e^x . e^{yi})} = (e^{\bar{x}} . e^{\overline{yi}})$$

$$= e^x . e^{-yi} = e^{x-yi} = e^{\bar{z}}$$

Example(15-3): Solve the equations

[1] $e^z = 1 + 0i$

Solution : $|e^z| = |1| = 1$

$$e^x = 1 \Longrightarrow lne^x = ln1$$

$$x = 0$$

$$\arg(e^z) = y + 2k\pi \quad , k \in z$$

$$y = arge^z + 2k\pi$$

$$y = \arg(1) + 2k\pi = tan^{-1}\left(\frac{0}{1}\right) + 2k\pi$$

$$y = 0 + 2k\pi = 2k\pi$$

$$z = x + yi = 2k\pi i \quad , \qquad k \in z$$

[2] $e^z = 1 + \sqrt{3}i$

Solution : $|e^z| = \left|1 + \sqrt{3}i\right|$

$$e^x = 2 \Longrightarrow lne^x = ln2$$

$$x = ln2$$

$$y = arge^z + 2k\pi$$

$$y = \arg\left(1 + \sqrt{3}i\right) + 2k\pi = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) + 2k\pi$$

$$= \frac{\pi}{3} + 2k\pi , k \in z$$

$$z = x + yi$$

$$z = ln2 + \left(\frac{1}{3} + 2k\right)\pi i \quad , \qquad k \in z$$

[3] $e^{4z} = i$

Solution : $|e^{4z}| = |\, i|$

$$e^{4x} = 1 \Rightarrow lne^{4x} = ln1$$

$$4x = 0$$

$$4y = arge^{4z} + 2k\pi$$

$$4y = \ tan^{-1}\left(\frac{1}{0}\right) + 2k\pi$$

$$4y = \frac{\pi}{2} + 2k\pi$$

$$y = \frac{\pi}{8} + 2k\pi$$

$$z = x + yi$$

$$z = \left(\frac{1}{8} + 2k\right)\pi i \quad , \qquad k \in z$$

[4] $2e^{z} = 4i$ (H.W)

- **The logarithm الدالة اللوغارتمية**

$$logz = ln|z| + i(Argz + 2k\pi) \quad , k \in z$$
$$= ln|z| + iargz$$

Example(15-4): find

[1] $\log(1)$

Solution : $ln|1| + i\,[Arg(1) + 2k\pi]$

$$= \ ln1 + i\left[tan^{-1}\left(\frac{0}{1}\right) + 2k\pi\right]$$

$$= 0 + i[0 + 2k\pi]$$

$$= 2k\pi i \quad , k \in z$$

[2] $\log(-1)$

Solution : $ln|-1| + i\ [Arg(-1) + 2k\pi]$

$$= \ ln1 + i\left[tan^{-1}\left(\frac{0}{-1}\right) + 2k\pi\right]$$

$$= 0 + i[\pi + 2k\pi]$$

$$= (1 + 2k)\pi i \quad , k \in z$$

[3] $\log\left(\sqrt{3} + i\right)$

Solution : $ln\left|\sqrt{3} + i\right| + i\ \left[Arg\left(\sqrt{3} + i\right) + 2k\pi\right]$

$$= \ ln2 + i\left[tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + 2k\pi\right]$$

$$= ln2 + i\left[\frac{\pi}{6} + 2k\pi\right]$$

$$= ln2 + \left(\frac{1}{6} + 2k\right)\pi i \quad , k \in z$$

[4] $\log(1 + i)$ (H.W)

Definition (15-1): The principal value of the logarithm of a nonzero complex number z , $logz$ is defined to be

$$Logz = ln|z| + iArgz$$

Example(15-5): find

[1] $\log(-1)$

Solution : $ln|-1| + i\,Arg(-1)$

$$= ln1 + itan^{-1}\left(\frac{0}{-1}\right)$$

$$= 0 + i\pi$$

$$= \pi i$$

[2] $\log(-1+i)$

Solution : $ln|-1+i| + i\,Arg(-1+i)$

$$= \ln\sqrt{2} + itan^{-1}\left(\frac{1}{-1}\right)$$

$$= \ln\sqrt{2} - i\,\frac{\pi}{4}$$

[2] $\log(-i)$ (H.W)

Remark(15-1):

[1] $Log\ z$ is a function

[2] $logz = Logz + 2k\pi i, \quad k \in z$

Example(15-6): solve the equations

[1] $logz = 1$

Solution :

$$e^{logz} = e^{1}$$

$$z = e$$

[2] $logz = \frac{\pi}{4}i$

Solution :

$$e^{logz} = e^{\frac{\pi}{4}i}$$

$$z = cos\frac{\pi}{4} + isin\frac{\pi}{4}$$

$$z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

[3] $logz = 1 + \frac{\pi}{3}i$ (H.W)

Proposition(15-2): Let z_1 and z_2 be two nonzero complex numbers , then

[1] $log(z_1.z_2) = logz_1 + logz_2$

[2] $log\left(\frac{z_1}{z_2}\right) = logz_1 - logz_2$

[3] $logz^n = nlogz$

[4] $\frac{d}{dz}[logz] = \frac{1}{z}$

Proof: [1]

$$log(z_1.z_2) = ln|z_1.z_2| + iarg(z_1.z_2)$$

$$= ln|z_1|.ln|z_2| + iarg(z_1.z_2)$$

$$= ln|z_1|.ln|z_2| + iargz_1 + iargz_2$$

$$= ln|z_1| + iargz_1.+ln|z_2| + iargz_2$$

$$= logz_1 + logz_2$$

Example(15-7): prove that $z^n = e^{nlogz}$ $,n \in z$

Solution :

$$e^{nlogz} = e^{n[ln|z| + i[Argz + 2k\pi i]]}$$

$$= e^{nln|z|}.e^{ni[Argz+2k\pi]}$$

$$= e^{\ln|z|^n}.e^{niArgz}.e^{2k\pi i}$$

$$= |z|^n \left(e^{iArgz}\right)^n .(1)$$

$$= \left[|z|. e^{iArgz}\right]^n = z^n$$

<u>Example(15-8):</u> Show that $logi^3 \neq 3Logi$ (H.W)

Complex analysis (lecture 14)

-
- **Harmonic conjugate المرافق التوافقي**

- المرافق التوافقي Harmonic conjugate

اذا كانت الدالتين $u(x,y)$ $v(x,y)$ توافقيتين بحيث ان الدالة $f(z) = u(x,y) + iv(x,y)$ تحليلية فان الدالة $v(x,y)$ تسمى المرافق التوافقي للدالة $u(x,y)$

خطوات حل اذا كان الجزء الحقيقي معلوم $u(x,y)$

1 – نثبت ان الدالة توافقية $u_{xx} + v_{yy} = 0$

2 - $v_y = u_x$ نكاملها بالنسبة الى y ونصنف دالة ولتكن $h(x)$ تكون ثابتة الى x

3 – نشتق ناتج رقم 2 بالنسبة الى x

4 - $v_x = -u_y$ نعوض وثم نجد $h'(x)$

5 – نكامل $h'(x)$ ونجد الناتج لكي نعوضه في ناتج رقم 2 حتى تكون $v(x,y)$

6 – نكتب الدالة $f(z) = u(x,y) + iv(x,y)$

خطوات حل اذا كان الجزء الحقيقي معلوم $v(x,y)$

1 – نثبت ان الدالة توافقية $v_{xx} + v_{yy} = 0$

2 - $u_y = -v_x$ نكاملها بالنسبة الى x ونصنف دالة ولتكن $g(y)$ تكون ثابتة الى y

3 – نشتق ناتج رقم 2 بالنسبة الى y

4 - $u_y = -v_x$ نعوض وثم نجد $g'(y)$

5 – نكامل $g'(y)$ ونجد الناتج لكي نعوضه في ناتج رقم 2 حتى تكون $u(x,y)$

6 – نكتب الدالة $f(z) = u(x,y) + iv(x,y)$

Example(14-1): construct an analytic function where real part is $u(x, y) = e^x \cos y + x^2 - y^2$

Find the harmonic conjugate $y(x, y)$

Solution :

1)

$$u_x = e^x \cos y + 2x \qquad u_{xx} = e^x \cos y + 2$$

$$u_y = -e^x \sin y - 2y \qquad u_{yy} = -e^x \cos y - 2$$

$$u_{xx} + u_{yy} = e^x \cos y + 2 + -e^x \cos y - 2 = 0$$

اذا الدالة توافقية

2)

$$v_y = u_x = e^x \cos y + 2x$$

$$v = \int (e^x \cos y + 2x) dy + h(x)$$

$$v = e^x \sin y + 2xy + h(x) \ldots \ldots .. (*)$$

3)

$$v_x = e^x \sin y + 2y + h'(x)$$

4)

$$e^x \sin y + 2y + h'(x) = e^x \sin y + 2y$$

$$h'(x) = 0 \Longrightarrow$$

5)

$$\int h'(x) = \int 0 dx$$

$$h(x) = 0 + c \quad \Longrightarrow h(x) = c \;\; in \; (*)$$

$$\therefore v = e^x siny + 2xy + c$$

$$\therefore f(z) = u(x,y) + iv(x,y)$$

$$f(z) = e^x cosy + x^2 - y^2 + i(e^x siny + 2xy + c)$$

$$= e^x cosy + x^2 - y^2 + ie^x siny + i2xy + ic$$

$$= [cosy + isiny] + x^2 - y^2 + 2ixy + ic$$

$$= e^x . e^{iy} + z^2 + ic$$

$$= e^{x+iy} + z^2 + ic$$

$$= e^z + z^2 + ic$$

Example(14-2): find out the analytic function having the real part as $u(x,y) = 2x(1-y)$

Solution :

1)

$$u_x = 2(1-y) \qquad u_{xx} = 0$$

$$u_y = 2x(-1) \qquad u_{yy} = 0$$

$$u_{xx} + u_{yy} = 0$$

اذا الدالة توافقية

2)

$$v_y = u_x = 2 - 2y$$

$$v = \int (2-2y)dy + h(x)$$

3)

$$v_x = h'(x)$$

4)

$$v_x = -u_y$$

$$h'(x) = -2x(-1) = 2x$$

5)

$$\int h'(x) = \int 2x dx$$

$$h(x) = x^2 + c \quad in\ (*)$$

$$\therefore v = 2y - y^2 + x^2 + c$$

$$\therefore f(z) = u(x,y) + iv(x,y)$$

$$f(z) = 2x(1-y) + i(2y - y^2 + x^2 + c)$$

$$= 2x - 2xy + 2yi - y^2 i + x^2 i + ic$$

$$= 2x + 2yi + (x^2 i - y^2 i - 2xy) + ic$$

$$= 2(x + yi) + i[x^2 - y^2 - 2ixy] + ic$$

$$= 2z + z^2 i + ic$$

Example(14-3): find out the analytic function having the imaginary parts as

[1] $v(x,y) = 2y(x+1)$ [2] $v(x,y) = 2xy$ (H.W)

Solution : [1]

1)

$$v_x = 2y \qquad v_{xx} = 0$$

$$v_y = 2(x+1) \qquad v_{yy} = 0$$

$$u_{xx} + u_{yy} = 0$$

اذا الدالة توافقية

2)

$$u_x = u_y = 2x + 2$$

$$u = \int (2x + 2)dx + g(x)$$

$$u = x^2 + 2x + g(y) \ldots \ldots . (*)$$

3)

$$u_y = g'(y)$$

4)

$$u_y = -\, v_x$$

$$g'(y) = -2y$$

5)

$$\int g'(y)dy = \int -2ydy$$

$$g(y) = -\, y^2 + c \quad in\ (*)$$

$$\therefore u = \; x^2 + 2x - y^2 + c$$

$$\therefore f(z) = u(x, y) + iv(x, y)$$

$$f(z) = x^2 + 2x - y^2 + c + i(2xy + 2y)$$

$$= x^2 - y^2 + 2ixy + 2x + 2yi + c$$

$$= z^2 + 2z \; + c$$

Complex analysis (lecture 13)

- **Entire function الدالة الكلية**
- **Singular point النقطة الشاذة**
- **Isolated point النقطة المعزولة**
- **Harmonic function الدوال التوافقية**

Entire function الدالة الكلية

Definition(13-1):

If a function $f(z)$ is analytic at each point the entire finite complex plane . then $f(z\ )$ is called an entire function .

Example(13-1): Show that $f(z) = \ cosxcoshy - isinxsinhy$

is an entire function .

Solution :

$$u \ = cosxcoshy \quad ,v \ = -sinxsinhy$$

$$u_x = -sinxcoshy \qquad v_x = -cosxsinhy$$

$$u_y = cosxsinhy \qquad v_y = -sinxcoshy$$

$$\therefore\ u_x = \ v_y \ \ and\ u_y = -\,v_x$$

$$f'(z) = \ -sinxcoshy - icosxsinhy$$

الدالة تحقق شروط كوشي ريمان

∴ f is entire function

Example(13-2): Let $f(z) = \ 2xy + i(x^2 - y^2)$ is an entire function?

Solution :

$$u \ = 2xy \quad ,v \ = x^2 - y^2$$

$$u_x = 2x \qquad v_x = 2x$$

$$u_y = 2y \qquad v_y = -2y$$

$$\therefore\ u_x \ \neq \ v_y \ \ and\ u_y \ \neq -\,v_x$$

الدالة لا تحقق شروط كوشي ريمان

$\therefore$ f is not entire function

Example(13-3): Let $f(z) = x^2 + y + i(y^2 - x)$ is an entire function? **(H.W)**

- Singular point النقطة الشاذة

Definition(13-2):

If a function $f(z)$ fails to be analytic at a point z_0 but is analytic at some point in every neighborhood of z_0 then z_0 is called a singular point .

تسمى نقطة شاذة للدالة اي ان النقطة الشاذة هي النقطة التي تجعل الدالة غير قابلة للاشتقاق و الجوار الذي يحوي النقطة الشاذة يسمى الجوار المحذوف .

Example(13-4): find the singular points of the functions

[1] $f(z) = \frac{1+z}{1-z}$

Solution :

$$1 - z = 0 \Longrightarrow z = 1$$

و هي النقطة الشاذة

[2] $f(z) = \frac{2z+1}{z(z^2+1)}$

Solution :

$$z(z^2 + 1) = 0$$

$$either\ z = 0$$

$$or\quad z^2 + 1 = 0 \Longrightarrow z^2 = -1 \Longrightarrow z = \mp i$$

الشاذة النقاط هي $0, i, -i$

[3] $f(z) = \frac{z^2+1}{(z+2)(z^2+2z+1)}$ (H.W)

Isolated point (منعزلة) النقاط المعزولة

If f is analytic in at least one deleted nbd.

تسمى النقطة معزولة اذا كانت z_0 نقطة شاذة وكان يوجد جوار الى z_0 تكون الدالة تحليلية في كل نقطة من نقاطه عدا z_0

Harmonic functions الدوال التوافقية

Let $h(x, y)$ be real function is called harmonic if

$$\frac{\partial^2 h}{\partial x} + \frac{\partial^2 h}{\partial y^2} = 0$$

the equation is called Laplas equation >

Example(13-5): prove that the function

$f(x, y) = e^x cosy$ is Harmonic function .

Solution :

$$\frac{\partial f}{\partial x} = e^x cosy \Longrightarrow \frac{\partial^2 f}{\partial x^2} = e^x cosy$$

$$\frac{\partial f}{\partial y} = -e^x siny \Longrightarrow \frac{\partial^2 f}{\partial y^2} = -e^x cosy$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^x cosy - e^x cosy = 0$$

هذا هي دالة توافقية

Example(13-6): prove that the function

$f(x, y) = 2x - x^3 + 3xy^2$ is Harmonic function

Solution :

$$\frac{\partial f}{\partial x} = 2 - 3x^2 + 3y^2 \Longrightarrow \frac{\partial^2 f}{\partial x^2} = -6x$$

$$\frac{\partial f}{\partial y} = 6xy \Longrightarrow \frac{\partial^2 f}{\partial y^2} = 6x$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -6x - 6x = 0$$

هذا هي دالة توافقية

Complex analysis (lecture 12)

-
- **Cauchy Riemain equations in polar coordinates معادلة كوشي ريمان بالصيغة القطبية**
- **Analytic function الدالة التحليلية**

Cauchy Riemain equations in polar coordinates معادلة كوشي ريمان بالصيغة القطبية

Definition(12-1):

If a function $f(z)$ is differentiable at a point z_0 , then there is a pair of equations called Cauchy Riemain equations must satisfied in Cartesian coordinates at a point z_0.

$$f(z) = u(r,\theta) + iv(r,\theta)$$

$$u_r = \frac{1}{r} v_\theta \ \ and\ v_r = -\frac{1}{r} u_\theta$$

Then

$$f'(z) = e^{-i\theta}[u_r + iv_r]$$

Example(12-1): $f(z) = \frac{1}{z}$

Is the function satisfy the Cauchy Riemain equations in polar coordinates ?

Solution :

$$f(z) = \frac{1}{r(cos\theta + isin\theta)} \cdot \frac{cos\theta - isin\theta}{cos\theta - isin\theta}$$

$$f(z) = \frac{cos\theta - isin\theta}{r}$$

$$f(z) = \frac{cos\theta}{r} - \frac{isin\theta}{r}$$

$$u(r,\theta) = \frac{cos\theta}{r} \qquad , v(r,\theta) = -\frac{sin\theta}{r}$$

$$u_r = -\frac{cos\theta}{r^2} \qquad v_r = \frac{sin\theta}{r^2}$$

$$u_{\theta} = -\frac{sin\theta}{r} \qquad v_{\theta} = -\frac{cos\theta}{r}$$

$$\therefore\ u_r = \frac{1}{r} v_{\theta}\ \ and\ v_r = -\frac{1}{r} u_{\theta}$$

الدالة تحقق شروط كوشي ريمان بالصيغة القطبية

$$f'(z) = e^{-i\theta}[u_r + iv_r]$$

$$f'(z) = e^{-i\theta}\left[-\frac{cos\theta}{r^2} + i\frac{sin\theta}{r^2}\right]$$

Example(12-2): Let $f(z) = \sqrt{r}e^{i\frac{\theta}{2}}$ Is the function satisfy the Cauchy Riemain equations in polar coordinates ?

Solution :

$$f(z) = \sqrt{r}\left[cos\frac{\theta}{2} + isin\frac{\theta}{2}\right]$$

$$f(z) = \sqrt{r}cos\frac{\theta}{2} + \sqrt{r}isin\frac{\theta}{2}$$

$$u(r,\theta) = \sqrt{r}cos\frac{\theta}{2} \qquad ,v(r,\theta) = \sqrt{r}isin\frac{\theta}{2}$$

$$u_r = \frac{cos\frac{\theta}{2}}{2\sqrt{r}} \qquad v_r = \frac{sin\frac{\theta}{2}}{2\sqrt{r}}$$

$$u_{\theta} = -\sqrt{r}\,sin\frac{\theta}{2}\,.\frac{1}{2} = -\frac{\sqrt{r}}{2}sin\frac{\theta}{2}$$

$$v_{\theta} = \sqrt{r}\,cos\frac{\theta}{2}\,.\frac{1}{2} = \frac{\sqrt{r}}{2}cos\frac{\theta}{2}$$

$$\therefore\ u_r = \frac{1}{r}v_{\theta} = \frac{\sqrt{r}\,cos\frac{\theta}{2}}{2r} = \frac{cos\frac{\theta}{2}}{2\sqrt{r}}$$

$$and\ v_r = -\frac{1}{r}u_\theta = -\frac{1}{r}\left(-\frac{\sqrt{r}}{2}\sin\frac{\theta}{2}\right)$$

الدالة تحقق شروط كوشي ريمان بالصيغة القطبية

$$f'(z) = e^{-i\theta}[u_r + iv_r]$$

$$f'(z) = e^{-i\theta}\left[\frac{\cos\frac{\theta}{2}}{2\sqrt{r}} + i\frac{\sin\frac{\theta}{2}}{2\sqrt{r}}\right]$$

- Analytic function الدالة التحليلية

Definition(12-2):

Suppose $f(z)$ is defined on a domain D , suppose z_0 is a point of D . Then f is said to be analytic at z_0 if f is differentiable not only at z_0 but also differentiable at each point of some $\epsilon - nbhd\ of\ z_0$.

Example(12-3): Show that $f(z) = \frac{1}{z}$ is analytic at z=1

Solution :

$$f'(z) = -\frac{1}{z^2}$$

$$z^2 = 0 \Longrightarrow z = 0$$

f' is exist $\forall\ z \in \mathbb{C} - \{0\}$

f' is exist $\forall\ z : |z-1| < \frac{1}{2}$

$\therefore f$ is analytic at z=1

Example(12-3): Show that $f(z) = |z|^2$ is not analytic at z=0

Solution :

$f(z) = x^2 + y^2$

$$u(x,y) = x^2 + y^2 \qquad , v(x,y) = 0$$

$$u_x = 2x \qquad v_x = 0$$

$$u_y = 2y \qquad v_y = 0$$

$$u_x \neq v_y \qquad u_y = -v_x$$

الدالة لا تحقق شروط كوشي ريمان

الدالة غير قابلة للاشتقاق

$$\therefore f \text{ } is \text{ } not \text{ } analytic \text{ } at \text{ } z = 0$$

Example(12-4): Let $f(z) = e^x[cosy + isiny]$

Is the function analytic function ?

Solution :

$$f(z) = cosy\, e^x + isiny\, e^x$$

$$u(x,y) = cosy\, e^x \qquad , v(x,y) = siny\, e^x$$

$$u_x = cosy\, e^x \qquad v_x = siny\, e^x$$

$$u_y = -\, e^x sinx \qquad v_y = e^x cosy$$

$$u_x = v_y \qquad u_y = -v_x$$

الدالة تحقق شروط كوشي ريمان

$$f'(z) = u_x + iv_x$$

$$f'(z) = e^x[cosy \ + \boldsymbol{i}siny \ ]$$

الدالة قابلة للاشتقاق

$$\therefore f \text{ } is \text{ } analytic \ \forall z \in \mathbb{C}$$

Example(12-5): Show that $f(z) = \frac{1}{(z-2)^2}$ is analytic in $S = \{z: |z| < 1\}$

Solution :

$$f'(z) = -\frac{2}{(z-2)^2}$$

$$(z-2)^2 = 0 \Rightarrow z = 2$$

f' is exist $\forall\, z \in \mathbb{C} - \{2\}$

f' is exist $\forall\, z \ \in S$

$\therefore f$ is analytic $\forall S$

Remark (12-1): A function f that is analytic in a set S which is not open it is to be understand that f ia analytic in an open set containing S.

Example(12-6): Show that $f(z) = \frac{1}{z}$ is analytic in

$S = \{z: |z-1| \ \leq \frac{1}{2}\}$

Solution :

$$f'(z) = -\frac{1}{z^2}$$

$$z = 0$$

f' is exist $\forall\, z \in \mathbb{C} - \{0\}$

f' is exist $\forall\, z \ \in S$

$\therefore f$ is analytic $\forall S$

Complex analysis (lecture 11)

- **Derivatives المشتقة**
- **Cauchy Riemain equation معادلة كوشي ريمان**
-

Derivatives :

Definition(11-1):

Let f be a function and let z_0 be an interior point of the domain of f . The derivative of f at z_0 written $f'(z)$ or $\frac{d}{dz}f(z)$

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z_0)}{\Delta z}$$

Example(11-1): Find $f'(z)$ if $f(z) = z^2 + z + i$

Solution :

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z_0)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 + (z + \Delta z) + i - (z^2 + z + i)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + \Delta z^2 + z + \Delta z + i - z^2 - z - i}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2z\Delta z + \Delta z^2 + \Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z[\, 2z + \Delta z + 1 \,]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 2z + \Delta z + 1 = 2z + 1$$

Example(11-2): Show that if $f(z) = Re(z)$ then $f'(z)$ does not exist.

Solution :

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z_0)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{Re(z+\Delta z) \ - \ Re(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{Re(z) + \ Re(\Delta z) - \ Re(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{Re(\Delta z)}{\Delta z}$$

$$let \quad \Delta z = \Delta x + i\Delta y$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{Re(\Delta x + i\Delta y)}{\Delta x + i\Delta y}$$

$$f'(z) = \begin{cases} \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1 \\ \lim_{\Delta y \to 0} \frac{0}{\Delta y} = 0 \end{cases}$$

$f'(z)$ does not exist.

<u>Example(11-3):</u> Show that if $f(z) = \bar{z}$ then $f'(z)$ does not exist.

<u>Solution :</u>

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z_0)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} \ - \ \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$let \quad \Delta z = \Delta x + i\Delta y$$

$$= \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$f'(z) = \begin{cases} \lim\limits_{\Delta x\to 0} \dfrac{\Delta x}{\Delta x} = 1 \\ \lim\limits_{\Delta y\to 0} \dfrac{-\Delta y}{\Delta y} = -1 \end{cases}$$

$f'(z)$ does not exist.

Example(11-3): if $f(z) = 2x + i3y$ show that $f'(z)$ does not exist at z_0. (H.W)

Definition(11-2):

If a function $f(z)$ is differentiable at a point z_0 , then there is a pair of equations called Cauchy Riemain equations must satisfied at a point z_0.

$$f(z) = u(x,y) + iv(x,y)$$

$$u_x = v_y \quad and \; u_y = -v_x$$

Example(11-4): $f(z) = x^2 - y^2 + i2xy$

Is the function satisfy the Cauchy Riemain equations ?

Solution :

$$u(x,y) = x^2 - y^2 \qquad , v(x,y) = 2xy$$

$$u_x = 2x \qquad v_x = 2y$$

$$u_y = -2y \qquad v_y = 2x$$

$$u_x = v_y \qquad u_y = -v_x$$

الدالة تحقق شروط كوشي ريمان

$$f'(z) = u_x + iv_x$$

$$f'(z) = 2x + i2y$$

Example(11-5): Let $f(z) = |z|^2$ Is the function satisfy the Cauchy Riemain equations ?

Solution :

Let $z = x + iy$

$$f(z) = x^2 + y^2 + 0i$$

$$u(x,y) = x^2 + y^2 \qquad ,v(x,y) = 0$$

$$u_x = 2x \qquad v_x = 0$$

$$u_y = 2y \qquad v_y = 0$$

$$u_x \neq v_y \qquad u_y \neq -v_x$$

الدالة لا تحقق شروط كوشي ريمان

Example(11-6): Let $f(z) = \bar{z}$ Is the function satisfy the Cauchy Riemain equations ?

Solution :

Let $z = x + yi \quad \Rightarrow \quad \bar{z} = x - yi$

$$f(z) = x - yi$$

$$u(x,y) = x \qquad ,v(x,y) = -y$$

$$u_x = 1 \qquad v_x = 0$$

$$u_y = 0 \qquad v_y = -1$$

$$u_x \neq v_y \qquad u_y \neq -v_x$$

الدالة لا تحقق شروط كوشي ريمان

Example(11-7): Let $f(z) = -sinx\ coshy - icosx\ sinhy$

Is the function satisfy the Cauchy Riemain equations ?

Solution :

$f(z) = -sinx\ coshy - icosx\ sinhy$

$$u(x,y) == -sinx\ coshy \qquad ,v(x,y) = -cosx\ sinhy$$

$$u_x = -coshy\ cosx \qquad v_x = sinhy\ sinx$$

$$u_y = -sinx\ sinhy \qquad v_y = -cosx\ coshy$$

$$u_x = v_y \qquad u_y = -v_x$$

الدالة تحقق شروط كوشي ريمان

$$f'(z) = u_x + iv_x$$

$$f'(z) = -coshy\ cosx + isinhy\ sinx$$

Example(11-8): Let $f(z) = \ e^x$

Is the function satisfy the Cauchy Riemain equations ?

Solution :

$$f(z) = cosy\ e^x + isiny\ e^x$$

$$u(x,y) = cosy\ e^x \qquad ,v(x,y) = siny\ e^x$$

$$u_x = cosy\ e^x \qquad v_x = siny\ e^x$$

$$u_y = -\ e^x sinx \qquad v_y = \ e^x cosy$$

$$u_x = v_y \qquad u_y = -v_x$$

الدالة تحقق شروط كوشي ريمان

$$f'(z) = u_x + iv_x$$

$$f'(z) = e^{x}[cosy \ \ + \boldsymbol{i}siny \ ]$$

Example(11-9): Let $f(z) = \ 3x + y + i(3y - x)$

Is the function satisfy the Cauchy Riemain equations ?

(H.W)

Complex analysis (lecture 10)

- **Continuity الاستمرارية**
-

Continuity

Let $f(z)$ be a function defined in some neighbourhood of the point z_0 . Then f is said to be continuous function at $z_0$iff $\lim_{z\to z_0} f(z) = f(z_0)$

That is , $f(z)$ is continuous at $z_0$iff

$$\forall\epsilon > 0 \ \exists\delta > 0 \ \ni |f(z) - f(z_0)| < \epsilon \ when \ |z - z_0| < \delta$$

<u>Example(10-1):</u> $f(z) = z^2$ is cont. function at the point $z_0 = 3$ since .

<u>Solution :</u>

$$\lim_{z\to 3} z^2 = 9$$

$$f(3) = 9$$

$$\lim_{z\to 3} f(z) = f(3)$$

$$\therefore f \ is \ cont.$$

<u>Example(10-2):</u> prove that $f(z) = z + 1$ is cont. function at the point $z_0 = 1$ by definition .

<u>Solution :</u>

$$\forall\epsilon > 0 \ \exists\delta > 0 \ \ni |f(z) - f(z_0)| < \epsilon \ when \ |z - z_0| < \delta$$

$$|z + 1 - f(1)| < \epsilon \ when \ |z - 1| < \delta$$

$$|z + 1 - 2| < \epsilon \ when \ |z - 1| < \delta$$

$$|z - 1| < \epsilon \ when \ |z - 1| < \delta$$

$$\epsilon = \delta$$

$$\therefore f \ is \ cont.$$

Example(10-3): Let $f(z) = \frac{z+i}{z-i}$ is f cont. ? at the point $z = 3i$ since .

Solution :

$$z - i = 0 \Rightarrow z = i$$

$$D: \mathbb{C} - \{i\}$$

$$\lim_{z \to 3i} \frac{z+i}{z-i} = \frac{3i+i}{3i-i} = \frac{4i}{2i} = 2$$

$$f(3i) = 2$$

$$\lim_{z \to 3i} f(z) = f(3i)$$

$$\therefore f \text{ is cont}$$

Example(10-4): Show that $f(z) = \frac{z^2 - iz + 4}{z}$ is f cont. at the point $z = i$.

Solution :

$$z = 0$$

$$D: \mathbb{C} - \{0\}$$

$$\lim_{z \to i} \frac{z^2 - iz + 4}{z} = \frac{i^2 - i.i + 4}{i} = \frac{-1 + 1 + 4}{i}$$

$$= \frac{4}{i} . \frac{-i}{-i} = -4i$$

$$f(i) = -4i$$

$$\lim_{z \to i} f(z) = f(i)$$

$$\therefore f \text{ is cont}$$

Example(10-5): Let $f(z) = \begin{cases} z^2 & ,z \neq i \\ 0 & ,z = i \end{cases}$ is f cont. at the point $z = i$? .

Solution :

$$\lim_{z \to i} z^2 = i^2 = -1$$

$$f(i) = 0$$

$$\lim_{z \to i} f(z) \neq f(i)$$

$$\therefore f \text{ is not cont at } z = i$$

Example(10-6): Let $f(z) = \begin{cases} \frac{z^2+1}{z+i} & ,z \neq -i \\ -2i & ,z = -i \end{cases}$ show that f is cont. at the point $z = -i$.

Solution :

$$\lim_{z \to -i} \frac{z^2+1}{z+i} = \frac{z^2 - i}{z+i}$$

$$\lim_{z \to -i} \frac{(z-i)(z+i)}{z+i} = -i - i = -2i$$

$$f(-i) = -2i$$

$$\lim_{z \to -i} f(z) = f(i)$$

$$\therefore f \text{ is cont at } z = -i$$

Theorem (10-1): A function $f(z) = u + iv$ is continuous at the point $z_0 = x_0 + y_0 i$ iff the functions u and v are continuous at the point (x_0, y_0).

Theorem (10-2): If $f(z)$ is continuous at the point z_0 , then $|f(z)|$ is also continuous at the point z_0.

Example(10-7): Let $f(z) = x^2 - y^2 + ixysinx$ show that f is cont. at the point $z = 0$.

Solution :

$$z = x + yi$$

$$\lim_{(x,y)\longrightarrow(0,0)} [(x^2 - y^2) + ixysinx]$$

$$= \lim_{(x,y)\longrightarrow(0,0)} (x^2 - y^2) + i \lim_{(x,y)\longrightarrow(0,0)} xysinx$$

$$= 0 + i0 = 0$$

$$f(0) = 0$$

$$\therefore f \text{ } is \text{ } cont$$

Example(10-8): Let $f(z) = \begin{cases} \frac{\bar{z}}{z} & , z \neq 0 \\ 0 & , z = 0 \end{cases}$ is f is cont. at the point $z = 0$? .

Solution :

$$\lim_{z\longrightarrow z_0} \frac{\bar{z}}{z}$$

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x - yi}{x + yi}$$

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x}{x} = 1$$

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{-y}{y} = -1$$

$$f(o) = 0$$

$$\therefore f \text{ } is \text{ } not \text{ } cont \text{ } at \text{ } z = o$$

Example(10-9): Let $f(z) = \begin{cases} \frac{xy}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$ is f is cont. at $(x,y) \neq (0,0)$? .

(H.W)

Complex analysis (lecture 9)

- $z_0 = x_0 + y_0$ الغاية تقترب من
- الغاية عند الما لانهاية
-

Theorem (9-1) : If $f(x) = u(x,y) + iv(x,y)$ then is defined at all points in some nbhd of a point $z_0 = x_0 + iy_0 \ and$

$w = w_1 + iw_2$ the :

$$\mathrm{Lim}_{z\to z_0} f(z) = w \iff \lim_{(x,y)\to(x_0,y_0)} u(x,y) = w_1$$

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = w_2$$

Example(9-1): Show that

$$\lim_{z\to 1-i}\big(x + i(2x+y)\big) = 1 + i \quad ,z = x + yi$$

Solution : $= \lim_{x+yi\to(1,-1)}\big(x + i(2x+y)\big)$

$$= 1 + i(2(1) - 1) = 1 + i$$

Example(9-2): If $f(z) = \frac{z}{z+\bar{z}}$ find $\lim_{z\to 1} f(z)$

Solution : Let $z = x + yi \ then \ \bar{z} = x - yi$

$$\lim_{z\to 1}\frac{z}{z+\bar{z}} = \lim_{x+yi\to(1,0)}\frac{x+yi}{x+yi+x-yi}$$

$$= \lim_{x+yi\to(1,0)}\frac{x+yi}{2\,x}$$

$$= \frac{1+0}{2(1)} = \frac{1}{2}$$

Example(9-3): Find $\lim_{z\to i}(i\bar{z} - 3)$

Solution : Let $z = x + yi \ then \ \bar{z} = x - yi$

$$\lim_{z\to i}(i\bar{z} - 3) = \lim_{x+yi\to(0,1)} (i(x - yi) - 3)$$

$$= \lim_{x+yi \to (0,1)} (y - 3 + xi)$$

$$= (1 - 3 + 0) = -2$$

Example(9-4): Find $\lim_{z \to i} \frac{1}{1-Re(z)}$

Solution : Let $z = x + yi \ \ then \ \bar{z} = x - yi$

$$\lim_{z \to i} \frac{1}{1 - Re(z)} = \lim_{x+yi \to (0,1)} \frac{1}{1 - x}$$

$$= \frac{1}{1 - 0} = 1$$

Example(9-5): Find $\lim_{z \to 1+i} \frac{z^2+z-3}{z+1}$

Solution : Let $z = x + yi$

$$\lim_{z \to 1+i} \frac{z^2 + z - 3}{z + 1} = \frac{(1 + i)^2 + (1 + i) - 3}{1 + i + 1}$$

$$= \frac{2i + i - 2}{2 + i} = \frac{-2 + 3i}{2 + i} \times \frac{2 - i}{2 - i}$$

$$= \frac{-4 + 2i + 6i + 3}{4 + 1} = \frac{-1}{5} + \frac{8}{5} i$$

Example(9-6): Find $\lim_{z \to i} \frac{z^3+i}{z-i}$ (H.W)

الغاية عند الما لانهاية

Remarks: $\frac{n}{\infty} = 0, \qquad \frac{n}{0} = \infty$

Example(9-7): Find the limits:

[1] $\lim_{z \to 1} \frac{1}{(z-1)^3}$

Solution : $= \frac{1}{(1-1)^3} = \frac{1}{0} = \infty$

[2] $\lim_{z\to 0}\frac{1}{z}$

Solution : $\lim_{z\to 0^+}\frac{1}{z}=\frac{1}{0}=\infty=L_1$ اليمين

$$\lim_{z\to 0^-}\frac{1}{z}=\frac{1}{0}=-\infty=L_2$$ اليسار

$$\therefore L_1\neq L_2$$

$$\therefore \lim_{z\to 0}\frac{1}{z}\ is\ not\ exist$$

[3] $\lim_{z\to 1^+}\frac{1}{z-1}$

Solution : $=\lim_{z\to 1^+}\frac{1}{z-1}=\frac{1}{0}=\infty$

[4] $\lim_{z\to\frac{-1}{3}}\frac{5z}{3z+1}$ (H.W)

[5] $\lim_{z\to 0^-}\frac{1}{z^3}$

Solution : $=\lim_{z\to 0^-}\frac{1}{z^3}=\frac{1}{0}=-\infty$

Example(9-8): Find the limits:

[1] $\lim_{z\to\infty}\frac{2z+3}{3z+2}$ $\Big]\div z$

Solution : $=\lim_{z\to\infty}\frac{\frac{2z}{z}+\frac{3}{z}}{\frac{3z}{z}+\frac{2}{z}}=\lim_{z\to\infty}\frac{2+\frac{3}{z}}{3+\frac{2}{z}}=\frac{2}{3}$

ملاحظة: عامل اعلى اس بالبسط على معامل اعلى اس بالمقام

1- في حالة تساوي الاس في المقام و البسط

[2] $\lim_{z\to\infty}\frac{1+6z}{z-2}$ $\Big]\div z$

Solution : $=\lim_{z\to\infty}\frac{\frac{1}{z}+\frac{6z}{z}}{\frac{z}{z}-\frac{2}{z}}=\lim_{z\to\infty}\frac{\frac{1}{z}+6}{\frac{2}{z}-1}=6$

[3] $\lim\limits_{z\to\infty} \dfrac{2z^2-7z+3}{3z^2+5z+1} \Big] \div z^2$

Solution : $= \lim\limits_{z\to\infty} \dfrac{\frac{2z^2-7z+3}{z^2}}{\frac{3z^2+5z+1}{z^2}} = \lim\limits_{z\to\infty} \dfrac{2-\frac{7}{z}+\frac{3}{z^2}}{3+\frac{5}{z}+\frac{1}{z^2}} = \dfrac{2}{3}$

[4] $\lim\limits_{z\to\infty} \dfrac{z^5-7}{-7z^5+1} \Big]$ $(H.w)$

2- اذا كانت درجة المقام اكبر من درجة البسط فنقسم على اكبر درجة موجودة في المقام

Example(9-9): Find the limits:

[1] $\lim\limits_{z\to\infty} \dfrac{z^3+7z+6}{z^5+2z^2+9} \Big] \div z^5$

Solution : $= \lim\limits_{z\to\infty} \dfrac{\frac{z^3+7z+6}{z^5}}{\frac{z^5+2z^2+9}{z^5}} = \lim\limits_{z\to\infty} \dfrac{\frac{1}{z^2}+\frac{7}{z^4}+\frac{6}{z^5}}{1+\frac{2}{z^3}+\frac{9}{z^5}} = \dfrac{0}{1} = 0$

[2] $\lim\limits_{z\to\infty} \dfrac{z}{2z^4+5} \Big] \div z^4$

Solution : $= \lim\limits_{z\to\infty} \dfrac{\frac{z}{z^4}}{\frac{2z^4}{z^4}} = \lim\limits_{z\to\infty} \dfrac{\frac{1}{z^3}}{2+\frac{5}{z^4}} = \dfrac{0}{2} = 0$

3- اذا كانت درجة البسط اكبر من درجة المقام فنقسم على اكبر درجة موجودة في المقام

Example(9-10): Find the limits:

[1] $\lim\limits_{z\to\infty} \dfrac{z^2+1}{z-1} \Big] \div z$

Solution : $= \lim\limits_{z\to\infty} \dfrac{\frac{z^2}{z}+\frac{1}{z}}{\frac{z}{z}-\frac{1}{z}} = \lim\limits_{z\to\infty} \dfrac{z+\frac{1}{z}}{1-\frac{1}{z}} = \dfrac{\infty}{1} = \infty$

[2] $\lim\limits_{z\to-\infty} \dfrac{3\,z^4+z+2}{z^2+z} \quad \Big]\div z^2$

Solution : $= \lim\limits_{z\to-\infty} \dfrac{\frac{3\,z^4+z+2}{z^2}}{\frac{z^2+z}{z^2}} = \lim\limits_{z\to\infty} \dfrac{\frac{z}{z^2}+\frac{3z^4}{z^2}+\frac{2}{z^2}}{\frac{z^2}{z^2}+\frac{z}{z^2}} = \dfrac{\infty}{1} = \infty$

4- في حالة الجذور

Example(9-11): Find the limits:

[1] $\lim\limits_{z\to\infty} \dfrac{\sqrt{2z^2+4}}{z-5} \quad \Big]\div z$

Solution : $= \lim\limits_{z\to\infty} \dfrac{\sqrt{\frac{2z^2}{z^2}+\frac{4}{z^2}}}{\frac{z}{z}-\frac{5}{z}} = \lim\limits_{z\to\infty} \dfrac{\sqrt{2+\frac{4}{z^2}}}{1-\frac{5}{z}} = \dfrac{\sqrt{2}}{1} = \sqrt{2}$

[2] $\lim\limits_{z\to\infty} \dfrac{3z}{\sqrt{4z^2+2}} \quad \Big]\div z$

Solution : $= \lim\limits_{z\to\infty} \dfrac{\frac{3z}{z}}{\sqrt{\frac{4z^2}{z^2}+\frac{2}{z^2}}} = \lim\limits_{z\to\infty} \dfrac{3}{\sqrt{4+\frac{2}{z^2}}} = \dfrac{3}{2}$

Abstruct:

[1] $\lim\limits_{z\to\infty} \dfrac{mz^a}{nz^a} = \dfrac{m}{n}$

[2] $\lim\limits_{z\to\infty} \dfrac{mz^a}{nz^b} = 0, \quad b > a$

[3] $\lim\limits_{z\to\infty} \dfrac{mz^a}{nz^b} = \infty, \quad a > b$

Complex analysis (lecture 8)

- **Limits by definition** الغاية النهاية باستخدام التعريف
- $z_0 = x_0 + y_0$ الغاية تقترب من

Definition(8-1) : suppose f is defined at all points in some nbhd of a point z_0 by the statement that $\lim_{z\to z_0} f(z) = w_0$

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \ni |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

Example(8-1): If $f(z) = \frac{iz}{2}$ is defined on $|z| < 1$ prove that

$\lim_{z\to z_0} \frac{iz}{2} = \frac{i}{2}$ by definition .

Solution :

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \ni |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

$$\left|\frac{iz}{2} - \frac{i}{2}\right| < \epsilon \quad , |z - 1| < \delta$$

$$\left|\frac{iz}{2} - \frac{i}{2}\right| < \epsilon \quad \implies \frac{|i||z - 1|}{2} < \epsilon$$

$$\implies \frac{|z - 1|}{2} < \epsilon \implies |z - 1| < 2\epsilon$$

Example(8-2): If $f(z) = \frac{iz^2}{2}$ is defined on $|z| < 1$ prove that

$\lim_{z\to 1} \frac{iz^2}{2} = \frac{i}{2}$ by definition .

Solution :

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \ni |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$$

$$\left|\frac{iz^2}{2} - \frac{i}{2}\right| < \epsilon \quad , |z - 1| < \delta$$

$$\left|\frac{iz^2}{2} - \frac{i}{2}\right| < \epsilon \quad \implies \frac{|i||z^2 - 1|}{2} < \epsilon$$

$$\implies \frac{|z - 1||z + 1|}{2} < \epsilon \implies |z - 1| < \;\epsilon$$

$$\therefore \delta = \epsilon$$

Example(8-3): prove that $\lim_{z\to 2i} z^2 = -4$ by definition .

Solution :

$$\forall\epsilon > 0 \ \exists\delta > 0 \ \ni |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$$

$$|z^2 + 4| < \epsilon \quad , |z - 2i| < \delta$$

$$|z^2 + 4| < \epsilon \quad \Rightarrow |(z - 2i)(z + 2i)| < \epsilon$$

$$\Rightarrow |z - 2i||z + 2i| < \epsilon$$

$$\Rightarrow |z - 2i||4i| < \epsilon$$

$$\Rightarrow |z - 2i| < \frac{\epsilon}{4}$$

Example(8-4): prove that $\lim_{z\to i+1} (2z - 3i) = (2 - i)$ by definition .

Solution :

$$\forall\epsilon > 0 \ \exists\delta > 0 \ \ni |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$$

$$|(2z - 3i) - (2 - i)| < \epsilon \quad , |z - (1 + i)| < \delta$$

$$|2z - 3i - 2 + i| < \epsilon \quad \Rightarrow |2z - 2 - 2i| < \epsilon$$

$$\Rightarrow \left|2\big(z - (1 + i)\big)\right| < \epsilon$$

$$\Rightarrow 2|z - (1 + i)| < \epsilon$$

$$\Rightarrow |z - (1 + i)| < \frac{\epsilon}{2}$$

$$\therefore \delta = \frac{\epsilon}{2}$$

Example(8-5): prove that $\lim_{z\to i+1} \frac{z-1}{z-2} = 2$ by definition .

Solution :

$$\forall\epsilon > 0 \ \exists\delta > 0 \ \ni |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$$

$$\left|\frac{z-1}{z-2}-2\right|<\epsilon \quad ,|z-3|<\delta$$

$$\left|\frac{z-1-2z+4}{z-2}\right|<\epsilon \quad \Rightarrow \left|\frac{3-z}{z-2}\right|<\epsilon$$

$$\Rightarrow \frac{|-(z-3)|}{|z-2|}<\epsilon$$

$$\Rightarrow \frac{|z-3|}{1}<\epsilon$$

$$\Rightarrow |z-3|<\epsilon$$

$$\therefore \delta=\epsilon$$

Example(8-6): prove that $\lim_{z\to -i}\ \frac{z^2+1}{z+i}=-2i$ by definition

(H.W)

Example(8-7): prove that $\lim_{z\to i+1}\ x+\ (x+2y)=\ 2i$ by definition .

Solution :

$$\forall\epsilon>0\ \ \exists\delta>0\ \ \ni |z-z_0|<\delta\ \Rightarrow |f(z)-w_0|<\epsilon$$

$$|x+\ (x+2y)-\ 2i|<\epsilon \quad ,|z-i|<\delta$$

$$|x+xi+2yi-2i|<\epsilon \quad \Rightarrow |x+yi-i+(x+y-1)i|<\epsilon$$

$$\Rightarrow |x+(y-1)i+(x+y-1)i|\ <\epsilon$$

$$\Rightarrow\ |x+y-1|+|x+y-1|\ <\epsilon$$

$$\Rightarrow 2|x+y-1|<\ \epsilon$$

$$\Rightarrow |x+y-1|<\frac{\epsilon}{2}$$

$$|z-i|<\delta\ \ \Rightarrow |x+y-1|<\delta$$

$$\therefore \delta=\ \epsilon$$

Limits exist

Example(8-8): Find which of the following limits exist .

[1] $\lim_{z \to 0} \frac{z}{\bar{z}}$

Solution : Let $z = x + yi$ then $\bar{z} = x - yi$

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{x+yi \to 0} \frac{x+yi}{x-yi}$$

$$\lim_{x \to 0} \frac{x+yi}{x-yi} = \frac{0+yi}{0-yi} = \frac{yi}{-yi} = -1 = L_1$$

$$\lim_{y \to 0} \frac{x+yi}{x-yi} = \frac{x+0}{x-0} = \frac{x}{x} = 1 = L_2$$

$$\therefore L_1 \neq L_2$$

$$\therefore \lim_{z \to 0} f(z) dose\ not\ exist$$

[2] $\lim_{z \to 0} \frac{z^2}{|z|^2}$

Solution : Let $z = x + yi$

$$\lim_{z \to 0} \frac{z^2}{|z|^2} = \lim_{x+yi \to 0} \frac{(x+yi)^2}{|x+yi|^2} = \lim_{x+yi \to 0} \frac{x^2+2xyi-y^2}{x^2+y^2}$$

$$\lim_{x \to 0} \frac{x^2+2xyi-y^2}{x^2+y^2} = \frac{-y^2}{y^2} = -1 = L_1$$

$$\lim_{y \to 0} \frac{x^2+2xyi-y^2}{x^2+y^2} = \frac{x^2}{x^2} = 1 = L_2$$

$$\therefore L_1 \neq L_2$$

$$\therefore \lim_{z \to 0} f(z) dose\ not\ exist$$

[3] $\lim_{z\to 0}\left(\frac{\bar{z}}{z}\right)^2$ (H,W)

[4] $\lim_{z\to 1}\frac{1-\bar{z}}{1-z}$

Solution : Let $z = x + yi$ then $\bar{z} = x - yi$

$$\lim_{z\to 1}\frac{1-\bar{z}}{1-z} = \lim_{x+yi\to 1}\frac{1-x+yi}{1-x-yi}$$

$$\lim_{x\to 1}\frac{1-x}{1-x} = 1 = L_1$$

$$\lim_{y\to 1}\frac{yi}{-yi} = -1 = L_2$$

$$\therefore L_1 \neq L_2$$

$$\therefore \lim_{z\to 1} f(z) dose\ not\ exist$$

[5] $\lim_{z\to 0}\frac{\bar{z}^2-z^2}{z}$

Solution : Let $z = x + yi$ then $\bar{z} = x - yi$

$$\lim_{z\to 0}\frac{\bar{z}^2-z^2}{z} = \lim_{x+yi\to 0}\frac{(x-yi)^2-(x+yi)^2}{x+yi}$$

$$= \lim_{x+yi\to 0}\frac{x^2-2xyi-y^2-x^2-2xyi+y^2}{x+yi}$$

$$= \lim_{x+yi\to 0}\frac{-4xyi}{x+yi}$$

$$\lim_{x\to 0}\frac{-4xyi}{x+yi} = 0 = L_1$$

$$\lim_{y\to 0}\frac{-4xyi}{x+yi} = 0 = L_2$$

$$\therefore L_1 = L_2$$

$$\therefore \lim_{z \to 0} f(z) \textit{is exist}$$

Complex analysis (lecture 7)

- **Functions of a complex variable الدوال المركبة**
- **Real and Imaginary parts of a function**

Definition(7-1) : Let S be a set of complex numbers. A function f defined on S is a rule that assigns each $z \in S$ a single complex number w . In such case we write $w = f(z)$.

Example(7-1): Find the Domain if

[1] $f(z) = z^2 + 3z + 2$

Solution : $D: \mathbb{C}$

[2] $f(z) = \frac{3}{z-3}$

Solution : $z - 3 = 0 \quad \Rightarrow z = 3$

$$D: \mathbb{C} - \{3\}$$

[3] $f(z) = \frac{1}{z^2+1}$

Solution : $z^2 + 1 = 0 \quad \Rightarrow z = \mp\sqrt{-1}$

$$z = \mp i$$

$$D: \mathbb{C} - \{i, -i\}$$

- Real and Imaginary parts of a function

Let $w = f(z)$ *and* $z = x + yi$ be a function and let $w = f(z) = u(x, y) + iv(x, y)$ Then

$$f(z) = f(x + yi) = u + iv.$$

If $z = re^{i\theta}$ then $f(z) = u(r, \theta) + iv(r, \theta)$ in polar coordinates.

Example(7-2): Write $f(z) = z^2$ in the forms

[1] $f(z) = u(x, y) + iv(x, y)$

[2] $f(z) = u(r, \theta) + iv(r, \theta)$

Solution : $Let\ z = x + yi$

$$f(z) = f(x+yi) = (x+yi)^2$$

$$= x^2 + 2xyi - y^2$$

$$= x^2 - y^2 + 2xyi$$

$$z = re^{i\theta}$$

$$f\left(re^{i\theta}\right) = \left(re^{i\theta}\right)^2$$

$$= r^2 . e^{2i\theta}$$

$$= r^2[cos2\theta + isin2\theta]$$

$$= r^2cos2\theta + r^2isin2\theta$$

Example(7-3): Write the functions in the form

$f(z) = u(r,\theta) + iv(r,\theta)$

[1] $f(z) = 2z^2 - 3z + i$

Solution : $Let\ z = x + yi$

$$f(z) = 2(x+yi)^2 - 3(x+yi) + i$$

$$= 2[x^2 + 2xyi - y^2] - 3x - 3yi + i$$

$$= (2x^2 - 2y^2 - 3x) + (4xy - 3y + 1)i$$

[2] $f(z) = 2i\bar{z}$

Solution : $Let\ z = x + yi$

$$\bar{z} = x - yi$$

$$f(z) = 2i(x - yi)$$

$$= 2xi - 2yi^2$$

$$= 2xi + 2y$$

$$= 2y + 2xi$$

Example(7-4): Write the functions in the form

$f(z) = u(r,\theta) + iv(r,\theta)$

[1] $f(z) = |z|$

Solution : $Let\ z = re^{i\theta}$

$$f(z) = |re^{i\theta}|$$

$$= |r|.|e^{i\theta}|$$

$$= r|cos\theta + isin\theta| = r\sqrt{cos^2\theta + sin^2\theta}$$

$$r + 0i$$

[2]$f(z) = \frac{1}{z+1}$

Solution : $Let\ z = re^{i\theta}$

$$f(z) = \frac{1}{re^{i\theta}+1} \cdot \frac{re^{-i\theta}+1}{re^{-i\theta}+1}$$

$$= \frac{re^{-i\theta}+1}{r^2 + re^{i\theta} + re^{-i\theta} + 1}$$

$$= \frac{re^{-i\theta}+1}{r^2 + 1 + 2\left[\frac{e^{i\theta}+e^{-i\theta}}{2}\right]}$$

$$= \frac{re^{-i\theta}}{r^2 + 1 + 2rcos\theta} + \frac{1}{r^2 + 1 + 2rcos\theta}$$

Complex analysis (lecture 6)

- **Roots of complex numbers قوى وجذور الاعداد المركبة**

Roots of complex numbers

Let $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left[\frac{\theta+2k\pi}{n}\right]} = r^{\frac{1}{n}}\left[\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right]$

$$k = 0,1,\ldots,n-1$$

Example (6-1) : Find out the roots of $(-1)^{\frac{1}{2}}$

Solution : $z = -1 + 0i$

$$r = \sqrt{1+0} = 1$$

$$\theta_1 = \tan^{-1}\left(\frac{0}{-1}\right) = \tan^{-1}(0) = \pi$$

$$z = r\,[\cos\theta + i\sin\theta]$$

$$z = \cos\pi + i\sin\pi$$

$$z^{\frac{1}{2}} = \cos\frac{\pi+2k\pi}{2} + i\sin\frac{\pi+2k\pi}{2}$$

$$k = 0,1$$

$$where\ k = 0$$

$$z_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$z_1 = 0 + i = i$$

$$where\ k = 1$$

$$z_2 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$

$$z_2 = 0 - i = -i$$

$$S = \{-i, i\}$$

Example (6-2) : Find out the cube roots of $(-8i)$

Solution : $z = 0 - 8i$

$$r = \sqrt{0 + 64} = 8$$

$$\theta_1 = tan^{-1}\left(\frac{-8}{0}\right) = \frac{-\pi}{2}$$

$$z = r\ [cos\theta + isin\theta]$$

$$z = 8\ \left[\cos\frac{\pi}{2} - isin\frac{\pi}{2}\right]$$

$$z^{\frac{1}{3}} = \ 8^{\frac{1}{3}}\left[cos\frac{\frac{\pi}{2} + 2k\pi}{3} + isin\frac{\frac{\pi}{2} + 2k\pi}{3}\right]$$

$$k = 0,1,2$$

$$where\ k = 0$$

$$z_1 = 2\left[cos\frac{\pi}{6} - isin\frac{\pi}{6}\right]$$

$$z_1 = 2\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = \ \sqrt{3} - i$$

$$where\ k = 1$$

$$z_2 = 2\left[cos\frac{\frac{5\pi}{2}}{3} - isin\frac{\frac{5\pi}{2}}{3}\right]$$

$$z_2 = 2\left[cos\frac{5\pi}{6} - isin\frac{5\pi}{6}\right]$$

$$z_2 = 2\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = \ -\sqrt{3} - i$$

$$where\ k = 2$$

$$z_3 = 2\left[cos\frac{\frac{9\pi}{2}}{3} - isin\frac{\frac{9\pi}{2}}{3}\right]$$

$$z_3 = 2\left[cos\frac{3\pi}{2} - isin\frac{3\pi}{2}\right]$$

$$z_3 = 2[\,0 + i] = \,2i$$

$$S = \{\,2i, \sqrt{3} - i, -\sqrt{3} - i\}$$

Example (6-3) : Solve the equation $x^3 + i = 0$,

Solution : $x^3 + i = 0 \Longrightarrow \; x^3 = -i$

$$x^3 = cos\frac{-\pi}{2} + isin\frac{-\pi}{2}$$

$$x^3 = cos\frac{\pi}{2} - isin\frac{\pi}{2}$$

$$k = 0,1,2$$

$$where\ k = 0$$

$$x_1 = \; cos\frac{\frac{\pi}{2} + 2(0)\pi}{3} - isin\frac{\frac{\pi}{2} + 2(0)\pi}{3}$$

$$x_1 = cos\frac{\pi}{6} - isin\frac{\pi}{6}$$

$$x_1 = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$where\ k = 1$$

$$x_2 = cos\frac{\frac{\pi}{2} + 2\pi}{3} - isin\frac{\frac{\pi}{2} + 2\pi}{3}$$

$$x_2 = cos\frac{5\pi}{6} - isin\frac{5\,\pi}{6}$$

$$x_2 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$where\ k = 2$$

$$x_3 = cos\frac{9\pi}{6} - isin\frac{9\pi}{6}$$

$$x_3 = 0 - i = i$$

$$S = \{\ i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i\}$$

Example (6-4) : Find out the roots of $\left(\sqrt{3} + i\right)^{\frac{-3}{2}}$

Solution : $z = \sqrt{3} + i$

$$r = \sqrt{3 + 1} = 2$$

$$\theta_1 = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z = r\ [cos\theta + isin\theta]$$

$$z = 2\ \left[\cos\frac{\pi}{6} - isin\frac{\pi}{6}\right]$$

$$z^{-3} = 2^{-3}\left[cos\frac{\pi}{6} + isin\frac{\pi}{6}\right]^{-3}$$

$$= \frac{1}{8}\left[cos\frac{\pi}{2} + isin\frac{\pi}{2}\right]$$

$$z^{\frac{-3}{2}} = \left(\frac{1}{8}\right)^{\frac{1}{2}}\left[cos\frac{\frac{\pi}{2} + 2k\pi}{2} - isin\frac{\frac{\pi}{2} + 2k\pi}{2}\right]$$

$$k = 0,1$$

$$where\ k = 0$$

$$z_1 = \frac{1}{2\sqrt{2}}\left[cos\frac{\pi}{4} - isin\frac{\pi}{4}\right]$$

$$z_1 = \frac{1}{2\sqrt{2}}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right] = \frac{1}{4} - \frac{1}{4}i$$

$$where\ k = 1$$

$$z_2 = \frac{1}{2\sqrt{2}}\left[cos\frac{5\pi}{4} - isin\frac{5\pi}{4}\right]$$

$$z_2 = \frac{1}{2\sqrt{2}}\left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right]$$

$$z_2 = -\frac{1}{4} + \frac{1}{4}i$$

$$S = \{-\frac{1}{4} + \frac{1}{4}i, \frac{1}{4} - \frac{1}{4}i\}$$

Complex analysis (lecture 5)

- **Eulers formula صيغة اويلر**
- **Demoivres formula صيغة دي موفر**

Products and quotients in exponential form

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ Then :

[1] $z_1.z_2 = r_1.r_2 e^{i(\theta_1+\theta_2)}$

[2] $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\vartheta_2)}$

[3] $\frac{1}{z} = \frac{1}{r} e^{-i\theta}$ H.W

Proof: [1] $z_1 = r_1 e^{i\theta_1} = r_1[cos\theta_1 + isin\theta_1]$

$$z_2 = r_2 e^{i\theta_2} = r_2[cos\theta_2 + isin\theta_2]$$

$$z_1.z_2 = r_1.r_2[cos\theta_1 cos\theta_2 + icos\theta_1 sin\theta_2 + isin\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2]$$

$$= r_1.r_2[(cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2) + i(cos\theta_1 sin\theta_2 + sin\theta_1 cos\theta_2)]$$

$$= r_1.r_2[cos(\theta_1+\theta_2) + isin(\theta_1+\theta_2)]$$

$$= r_1.r_2 e^{i(\theta_1+\theta_2)}$$

[2] $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} . \frac{e^{-i\theta_2}}{e^{-i\theta_2}}$

$$\frac{r_1}{r_2} e^{i\theta_1 - i\vartheta_2} = \frac{r_1}{r_2} e^{i(\theta_1-\vartheta_2)}$$

Example (5-1) : Let $z_1 = 1 - i$ and $z_2 = 1 + \sqrt{3}i$ find using Eulers formula :

[1] $z_1.z_2$ [2] $\frac{z_1}{z_2}$ [3] $\frac{1}{z_1}$ H.W

Solution : $z_1 = 1 - i$

$$r_1 = \sqrt{1+1} = \sqrt{2}$$

$$\theta_1 = tan^{-1}(-1) = -tan^{-1}(1) = \frac{-\pi}{4}$$

$$z_1 = \sqrt{2}e^{-\frac{\pi}{4}i}$$

$$z_2 = 1 + \sqrt{3}i$$

$$r_2 = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta_2 = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$z_2 = 2e^{\frac{\pi}{3}i}$$

[1] $z_1.z_2 = r_1.r_2e^{i(\theta_1+\theta_2)}$

$$= 2\sqrt{2}e^{i(-\frac{\pi}{4}+\frac{\pi}{3})}$$

$$= 2\sqrt{2}e^{i\frac{\pi}{12}}$$

[2] $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\vartheta_2)}$

$$= \frac{\sqrt{2}}{2}e^{i(-\frac{\pi}{4}-\frac{\pi}{3})}$$

$$= \frac{1}{\sqrt{2}}e^{i\frac{-7\pi}{12}}$$

<u>Example (5-2) :</u> Show that $sin^3\theta = \frac{3}{4}sin\theta - \frac{1}{4}sin3\theta$ using Eulers formula

<u>Solution :</u> $e^{i\theta} = cos\theta + isin\theta$

$e^{-i\theta} = cos\theta - isin\theta$ بالجمع

$e^{i\theta} + e^{-i\theta} = 2cos\theta \quad] \div 2$

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$e^{i\theta} = cos\theta + isin\theta$$

$$- e^{-i\theta} = -cos\theta + isin\theta$$ بالطرح

$$e^{i\theta} - e^{-i\theta} = 2sin\theta \quad] \div 2i$$

$$sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

الطرف الايسر

$$sin^3\theta = \left[\frac{e^{i\theta} - e^{-i\theta}}{2i}\right]^3 = \frac{\left[e^{i\theta} - e^{-i\theta}\right]^3}{2^3 . i^3}$$

$$= \frac{e^{3i\theta} - 3e^{2i\theta}e^{-i\theta} + 3e^{i\theta}\, e^{-2i\theta} - e^{-3i\theta}}{-8i}$$

$$= \frac{e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}}{-8i}$$

$$= \frac{e^{3i\theta} - e^{-3i\theta}}{-8i} - \frac{3e^{i\theta} - 3e^{-i\theta}}{-8i}$$

$$= -\frac{1}{4}sin3\theta + \frac{3}{4}sin\theta$$

$$= \frac{3}{4}sin\theta - \frac{1}{4}sin3\theta$$

Demoivres formula :

Let $z = r(cos\theta + isin\theta)$ Then

$$z^n = [r(cos\theta + isin\theta)]^n = r^n[cosn\theta + isinn\theta]$$
$$= r^n r^{in\theta} \quad ; n \in z$$

Example (5-3) : Write $\left(\sqrt{3} + i\right)^7$ in the form $x + yi$

<u>Solution :</u>

$$z = \sqrt{3} + i$$

$$r = \sqrt{x^2 + y^2} = \sqrt{3+1} = \sqrt{4 = 2}$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z = r[cos\theta + isin\theta]$$

$$z = 2\left[cos\frac{\pi}{4} + isin\frac{\pi}{4}\right]$$

$$z^7 = 2^7\left[cos\frac{7\pi}{6} + isin\frac{7\pi}{6}\right]$$

$$= 128\left[-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right]$$

$$= -64\sqrt{3} - 64i$$

<u>Example (5-4) :</u> Write $z = \left[\frac{-1+\sqrt{3}i}{\sqrt{3}-i}\right]$ in the form $x + yi$ H.W

<u>Example (5-5) :</u> prove that by Demoivers formula

[1] $cos2\theta = cos^3\theta - sin^2\theta$

[2] $sin2\theta = 2sin\theta cos\theta$

[3] $cos3\theta = 4cos^3\theta - 3cos\theta$

[4] $sin3\theta = 3sin\theta - 4sin^3\theta$

<u>Solution:</u> [1][2]

$$(cos\theta + isin\theta)^2 = cos^2\theta + 2isin\theta cos\theta + i^2sin^2\theta$$

$$cos2\theta + isin2\theta = cos^2\theta - sin^2\theta + 2isin\theta cos\theta$$

[2] $sin2\theta = 2sin\theta cos\theta$

<u>Solution:</u> [3][4]

$$(cos\theta + isin\theta)^3$$
$$= cos^3\theta + 3cos^2\theta . isin\theta + 3cos\theta . i^2 sin^2\theta + i^3 sin^3\theta$$

$$(cos3\theta + isin3\theta)$$
$$= cos^3\theta + 3icos^2\theta . sin\theta - 3cos\theta . sin^2\theta - isin^3\theta$$

$$\therefore cos3\theta = cos^3\theta - 3sin^2\theta cos\theta$$

$$= cos^3\theta - 3(1 - cos^2\theta)cos\theta$$

$$= cos^3\theta - 3cos\theta + 3cos^3\theta$$

$$\therefore cos3\theta = 4cos^3\theta - 3cos\theta$$

$$sin3\theta = 3cos^2\theta sin\theta - sin^3\theta$$

$$= 3(1 - sin^3\theta)sin\theta - sin^3\theta$$

$$= 3sin\theta - 3sin^3\theta - \ sin^3\theta$$

$$= 3sin\theta - 4sin^3\theta$$

Complex analysis (lecture 4)

- **Polar coordinates الصيغة القطبية**
- **Eulers formula صيغة اويلر**

Polar coordinates الصيغة القطبية

Let $r\ and\ \theta$ be polar coordinates of the point (x, y) that corresponds to a non-zero complex number $z = x + yi\ \ sice\ x = rcos\theta\ and\ y = rsin\theta$

z can be written polar form as $z = r[cos\theta + isin\theta]\ , r = |z| \ni$
$r = |z| = \sqrt{x^2 + y^2}\ and\ \theta = tan^{-1}\left(\frac{y}{x}\right)\ , \theta \in (-\pi, \pi]$,

$\theta = argz$. The principle value of arg z denoted by Argz then

$$Argz = argz + 2k\pi, \quad k = 0, \mp 1, \mp 2, \ldots.$$

Theorem (4-1) : Let $z\ and\ w$ be two complex numbers. Then

[1] $\arg(z.w) = argz + argw$

[2] $arg\left(\frac{z}{w}\right) = argz = argw$

[3] $\arg(\bar{z}) = -argz = arg\frac{1}{z}$

Proof:

[1] Let $z = |z|[\cos\theta_1 + isin\theta_1]$

$$w = |w|[\cos\theta_2 + isin\theta_2]$$

$$z.w = |z|.|w|[\cos\theta_1\cos\theta_2 + i\cos\theta_1 sin\theta_2 + isin\theta_1\cos\theta_2 - sin\theta_1 sin\theta_2]$$

$$= |z|.|w|[\cos\theta_1\cos\theta_2 - sin\theta_1 sin\theta_2 + i(sin\theta_2\cos\theta_1 + sin\theta_1\cos\theta_2)]$$

$$= |z|.|w|[\cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2)]$$

$$\therefore \arg(z.w) = argz + argw$$

Example (4-1): Find

[1] $Arg(3)$ [2] $Arg\ (-3)$ [3] $Arg(2i)$ [4]$Arg(-2i)$

[5] $Arg\left[\frac{1+i}{1-i}\right]$

Solution :

[1] $z = 3 + 0i$

$$Arg(3) = \ tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{0}{3}\right) = tan^{-1}(0) = 0 \in (-\pi, \pi]$$

[2] $z = -3 + 0i$

$$Arg(-3) \ = tan^{-1}\left(\frac{0}{-3}\right) = tan^{-1}(0) = \ \pi$$

[3] $0 + 2i$

$$Arg(2i) \ = tan^{-1}\left(\frac{2}{0}\right) \ = \frac{\pi}{2}$$

[4] $0 - 2i$

$$Arg(2i) \ = tan^{-1}\left(\frac{-2}{0}\right) \ = \frac{-\pi}{2}$$

[5] $z = \frac{1+i}{1-i} . \frac{1+i}{1+i}$

$$z = \frac{1 + 2i - 1}{2} = i$$

$$Arg\left(\frac{1+i}{1-i}\right) \ = tan^{-1}\left(\frac{1}{0}\right) \ = \frac{\pi}{2}$$

Example (4-2): Find $argz$ if

[1] $z = \sqrt{3} + i$ [2] $z = -1 - i$

Solution :

[1]

$$Arg(z) = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\therefore argz = Argz + 2k\pi \ , k \in z$$

$$argz = \frac{\pi}{6} + 2k\pi \ , k \in z$$

[2]

$$Arg(z) = tan^{-1}\left(\frac{-1}{-1}\right) = tan^{-1}(1) = \frac{\pi}{4}$$

$$Arg(z) = \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$$

$$\therefore argz = Argz + 2k\pi \ , k \in z$$

$$argz = \frac{-3\pi}{4} + 2k\pi \ , k \in z$$

Example (4-3): Find polar representation of

[1] $z = 1 + \sqrt{3}i$ [2] $z = 1 - i$

Solution :

[1] $r = \sqrt{x^2 + y^2} = \sqrt{1+3} = \sqrt{4} = 2$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\therefore z = r[cos\theta + isin\theta]$$

$$= 2\left[cos\frac{\pi}{3} + isin\frac{\pi}{3}\right]$$

[2] $r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{-1}{1}\right) = tan^{-1}(-1) = -tan^{-1}(1)$$
$$= \frac{-\pi}{4}$$

$$\therefore z = r[cos\theta + isin\theta]$$

$$= \sqrt{2}\left[cos\frac{-\pi}{4} + isin\frac{-\pi}{4}\right]$$

$$= \sqrt{2}\left[cos\frac{\pi}{4} - isin\frac{\pi}{4}\right]$$

Eulers formula الصيغة القطبية

Using the symbol $e^{i\theta}$ *or* $\exp(i\theta)$ which is defined by Eulers formula for any value of θ *as* $e^{i\theta} = cos\theta + isin\theta$

$$\therefore z = r[cos\theta + isin\theta] = re^{i\theta}$$

Example (4-3): if $z = -1$ find argument z and Eulers formula

Solution :

$$argz = Argz + 2k\pi \ \ ,k \in z$$

$$Arg(3) = \ tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{0}{-1}\right) = tan^{-1}(0) = \pi$$

$$argz = \pi + 2k\pi \ \ ,k \in z$$

$$r = |z| = |-1| = 1$$

$$\therefore z = re^{i\theta} = e^{\pi i}$$

Example (4-4): find polar formula

[1] $z = 1 + i$ [2] $z = \sqrt{3} - i$ [3] $z = -1 - i$ $H.W$

Solution :

[1] $r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{1}{1}\right) = tan^{-1}(1) = \frac{\pi}{4}$$

$$\therefore z = re^{i\theta} = \sqrt{2}e^{\frac{\pi}{4}i}$$

[2] $r = \sqrt{x^2 + y^2} = \sqrt{3+1} = \sqrt{4} = 2$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{-\pi}{6}$$

$$\therefore z = re^{i\theta} = 2e^{\frac{-\pi}{6}i}$$

<u>Example (4-5):</u> Express z in the form $x + yi$

[1] $z = 2e^{\frac{-\pi}{4}i}$ [2] $z = 3e^{\frac{\pi}{3}i}$ [3] $z = -5e^{\frac{5\pi}{6}i}$ $H.W$

<u>Solution :</u>

[1] $z = r[cos\theta + isin\theta]$

$$z = 2\left[cos\frac{-\pi}{4} + isin\frac{-\pi}{4}\right]$$

$$= 2\left[cos\frac{\pi}{4} - isin\frac{\pi}{4}\right]$$

$$= 2\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$= \sqrt{2} - \sqrt{2}i$$

[2]

$$z = 3\left[cos\frac{\pi}{3} + isin\frac{\pi}{3}\right]$$

$$= 3\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$$

$$= \frac{3}{2} + \frac{3\sqrt{3}}{2}\ i$$

Example (4-6): Show that $\left|e^{i\theta}\right| = 1$

Solution :

$z = r[cos\theta + isin\theta]$

$$z = \ cos\theta + isin\theta = e^{i\theta}$$

$$\therefore \left|e^{i\theta}\right| = |\ cos\theta + isin\theta| = \sqrt{cos^2\theta + sin^2\theta} = \sqrt{1} = 1$$

Example (4-7): Find $\left|e^{\frac{\pi}{2}i}\right|$

Solution :

$$\left|e^{\frac{\pi}{2}i}\right| = \left|\ cos\frac{\pi}{2} + isin\frac{\pi}{2}\right| = \ |0 + i(1)| = \ |i| = 1$$

Complex analysis (lecture 3)

- **Absolute value القيمة المطلقة**

Example : (3 -1) : Show that

[1] $|z_1 - z_2| \leq |z_1 - z_3| + |z_3 - z_2|$

Proof:

$$|z_1 - z_2| = |(z_1 - z_3) + (z_3 - z_2)|$$

$$\leq |z_1 - z_3| + |z_3 - z_2|$$

[2] $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Proof:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2$$

$$= (z_1 + z_2).\overline{(z_1 + z_2)} + (z_1 - z_2).\overline{(z_1 - z_2)}$$

$$= (z_1 + z_2).(\overline{z_1} + \overline{z_2}) + (z_1 - z_2).(\overline{z_1} - \overline{z_2})$$

$$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2}$$

$$= 2z_1\overline{z_1} + 2z_2\overline{z_2} = 2(|z_1|^2 + |z_2|^2)$$

[3] $\left|(2\bar{z} + 5)\left(\sqrt{2} - i\right)\right| = \sqrt{3}|2\bar{z} + 5|$

Proof:

$$|2\bar{z} + 5|\left|\sqrt{2} - i\right|$$

$$= |2\bar{z} + 5|\left(\sqrt{\left(\sqrt{2}\right)^2 + (-1)^2}\right)$$

$$= |2\bar{z} + 5|\sqrt{3}$$

$$= \sqrt{3}|2\bar{z} + 5|$$

Example : (3 -2) : If $iz^2 - \bar{z} = 0$ find values of $|z|$

Sol:

$$iz^2 - \bar{z} = 0$$

$$iz^2 = \bar{z} \implies |iz^2| = |\bar{z}|$$

$$|i||z^2| = |z|$$

$$|i| = |0 + i| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$$

$$|z^2| - |z| = 0 \implies |z|(|z| - 1) = 0$$

$$either\ |z| = 0 \quad or\ |z| - 1 = 0 \implies |z| = 1$$

Example : (3 -3) : If z lies on the circle $|z| = 2$ then show that r .

[1] $|z^2 + 2z - 1| \leq 9$

[2] $\frac{1}{|z^3-1|} \leq \frac{1}{7}$

[3] $3 \leq |z^2 - 1| \leq 5$

Sol:

[1] $|z^2 + 2z - 1|$

$$= |z^2 + 2z + (-1)|$$

$$\leq |z|^2 + 2|z| + 1$$

$$\leq 2^2 + 2(2) + 1 \leq 4 + 4 + 1 \leq 9$$

[2] $\frac{1}{|z^3-1|}$

$$|z^3 - 1| = |z^3 + (-1)|$$

$$\geq ||z|^3 - 1| \geq |8 - 1| \geq 7$$

$$n > k \implies \frac{1}{n} < \frac{1}{k}$$

$$\frac{1}{|z^3 - 1|} \leq \frac{1}{7}$$

[3] $3 \leq |z^2 - 1| \leq 5$

$$|z^2-1| = |z^2+(-1)|$$

$$\leq |z|^2 + |-1| \leq |z|^2 + 1$$

$$\leq 4+1 \leq 5$$

$\therefore\ |z^2-1| \leq 5$ طرف ايمن

$$|z^2-1| \geq 3$$

$$|z^2-1| = |z^2+(-1)|$$

$$\geq \big||z|^2 - |-1|\big|$$

$$\geq ||z|^2 - 1|$$

$\geq |4-1| \geq 3$ طرف ايسر

Example : (3 -4) : Prove that $\sqrt{2}\,|z| \geq |Re(z)| + |Im(z)|$

Proof: Let $z = x + yi$

$$\sqrt{2}\,|z| = \sqrt{2}\sqrt{x^2+y^2} = \sqrt{2x^2+2y^2}$$

$$(|x|-|y|)^2 \geq 0 \Longrightarrow |x|^2 + |y|^2 \geq 2|x||y|$$

$$\Longrightarrow\ x^2+y^2 \geq 2|x||y|$$

$$x^2+y^2+x^2-x^2+y^2-y^2 \geq 2|x||y|$$

$$2x^2+2y^2 \geq x^2 2|x||y| + y^2$$

$$2x^2+2y^2 \geq (|x|-|y|)^2 \Longrightarrow\ \sqrt{2}\sqrt{x^2+y^2} \geq |x|+|y|$$

Example : (3 -4) : If $|z_1| = 1\ or\ |z_2| = 1$ but not both then prove that $\left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right| = 1$

Proof:

$$\left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right| = \frac{|z_1-z_2|^2}{|1-\overline{z_1}z_2|^2}$$

$$= \frac{|z_1|^2 - 2|z_1||z_2| + |z_2|^2}{1 - 2|\overline{z_1}||z_2| + |\overline{z_1}\, z_2|^2}$$

$$= \frac{|z_1|^2 - 2|z_1||z_2| + |z_2|^2}{1 - 2|z_1||z_2| + |z_1|^2|z_2|^2}\ , \quad if\ |z_1| = 1$$

$$= \frac{1 - 2|z_2| + |z_2|^2}{1 - 2|z_2| + |z_2|^2} = 1$$

<u>Example : (3 -5) :</u> If $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$

Then find the value of $|z_1 + z_2 + z_3|$

Sol:

$$|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\overline{z_1} = 1 \Rightarrow \overline{z_1} = \frac{1}{z_1}$$

$$|z_2| = 1 \Rightarrow |z_2|^2 = 1 \Rightarrow z_2\overline{z_2} = 1 \Rightarrow \overline{z_2} = \frac{1}{z_2}$$

$$|z_3| = 1 \Rightarrow |z_3|^2 = 1 \Rightarrow z_3\overline{z_3} = 1 \Rightarrow \overline{z_3} = \frac{1}{z_3}$$

$$\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1 \Rightarrow |\overline{z_1} + \overline{z_2} + \overline{z_3}| = 1$$

$$\Rightarrow |z_1 + z_2 + z_3| = 1$$

<u>Example : (3 -6) :</u> Find the value of k if for the complex number $z_1$and z_2

$$|1 - \overline{z_1}\, z_2|^2 - |z_1 - z_2|^2 = k(1 - |z_1|^2)(1 - |z_2|^2)$$

Sol: L.H.S $|1 - \overline{z_1}\, z_2|^2 - |z_1 - z_2|^2$

$$= (1 - \overline{z_1}\, z_2)\overline{(1 - \overline{z_1}\, z_2)} - (z_1 - z_2)\overline{(z_1 - z_2)}$$

$$(1 - \overline{z_1}\, z_2)\overline{(1 - \overline{z_1}\, z_2)} - (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$

$$= 1 - z_1\overline{z_2} - \overline{z_1}z_2 + z_1\overline{z_1} - z_2\overline{z_2} - z_1\overline{z_1} + \overline{z_1}z_2 + z_1\overline{z_2} - z_2\overline{z_2}$$

$$= 1 + |z_1|^2|z_2|^2 - |z_1|^2 - |z_2|^2$$

$$= 1 - |z_1|^2 - |z_2|^2(1 - |z_1|^2) = (1 - |z_1|^2)(1 - |z_2|^2)$$

Example : (3 -7) : Show that $\left|\frac{z-2}{z^3}\right| = 2$ represents a circle . find its center and radius .

Sol:

Let $z = x + yi$

$$\left|\frac{x + yi - 2}{x + yi - 3}\right| = 2 \Longrightarrow \left|\frac{x - 2 + yi}{x - 3 + yi}\right| = 2$$

$$\frac{\sqrt{(x-2)^2 + y^2}}{\sqrt{(x-3)^2 + y^2}} = 2 \Longrightarrow \frac{(x-2)^2 + y^2}{(x-3)^2 + y^2} = 4$$

$$x^2 - 4x + 4 + y^2 = 4x^2 - 24x + 36 + 4y^2$$

$$3x^2 - 20x + 32 + 3y^2 = 0 \Longrightarrow 3x^2 + 3y^2 - 20x + 32 = 0 \} \div 3$$

$$x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$$x^2 - \frac{20}{3}x + \frac{100}{9} + y^2 + \frac{32}{3} - \frac{100}{9} = 0$$

$$\left(x - \frac{10}{3}\right)^2 + y^2 - \frac{9}{4} = 0$$

$$\left(x - \frac{10}{3}\right)^2 + y^2 = \frac{9}{4}$$

$$(x - h)^2 + (y - k)^2 = r^2$$

$\therefore$ the center is $(h,k) = \left(\frac{10}{3}, 0\right)$ and the radius is $\frac{2}{3}$

<u>Example : (3 -8) :</u> show that the equation $|z - 2 + i| = 3$ represents a circle its center $z_0 = 2 - i$ and radius 3.

Proof: Let $z = x + yi$

$$|z - 2 + i| = 3 \Longrightarrow |(x-2) + i(y+1)| = 3$$

$$\sqrt{(x-2)^2 + (y+1)^2} = 3$$

$$(x-h)^2 + (y-k)^2 = r^2$$

$$\therefore (h,k) = (2,-1) \Longrightarrow z_0 = 2 - i$$

$$r^2 = 9 \Longrightarrow r = 3$$

<u>Example : (3 -8) :</u> show that the equation $|z - z_0| = R$represents a circle its center z_0 and radius R can be written as .

$|z|^2 - 2Re(z\overline{z_0}) + |z_0|^2 = R^2$ H.W

Complex analysis (lecture 2)

- **Proof of some properties of complex conjugate composite preparation برهان بعض خواص العامل المرافق في الاعداد المركبة**

Definition : (2-1) : The complex conjugate or simply the conjugate of a complex number $z = x + yi$ is defined as the complex number

$\bar{z} = x - yi$

[1] $\bar{\bar{z}} = z$

Proof: if $z = x + yi$ then $\bar{z} = x - yi \Rightarrow \bar{\bar{z}} = x + yi = z$

[2] $|\bar{z}| = |z|$

Proof: if $z = x + yi$ then $\bar{z} = x - yi$

$$|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$$

[3] $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$

Proof : Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

$$\overline{z_1 + z_2} = \overline{(x_1 + y_1 \imath) + (x_2 + y_2 \imath)}$$

$$= \overline{(x_1 + x_2) + (y_1 + y_2)\imath}$$

$$= (x_1 + x_2) - (y_1 + y_2)i$$

$$= (x_1 + y_1) + (x_2 - y_2)i$$

$$= (x_1 - y_1 i) + (x_2 - y_2 i) = \bar{z_1} + \bar{z_2}$$

[4] $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$ H. W

[5] $\overline{z_1 . z_2} = \bar{z_1} . \bar{z_2}$

Proof : Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

$$\overline{z_1 . z_2} = \overline{(x_1 + y_1 \imath).(x_2 + y_2 \imath)}$$

$$= \overline{x_1 x_2 + x_1 y_2 \imath + x_2 y_1 - y_1 y_2}$$

$$= \overline{(x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)\imath}$$

$= (x_1 x_2 - y_1 y_2) - (x_1 y_2 + x_2 y_1)i$ طرف ايسر

$$\bar{z_1} . \bar{z_2} = \overline{(x_1 + y_1 \imath)}.\overline{(x_2 + y_2 \imath)}$$

$$= (x_1 - y_1 i).(x_2 - y_2 i)$$

$$= x_1x_2 - x_1y_2i - x_2y_1 - y_1y_2$$

$= (x_1x_2 - y_1y_2) - (x_1y_2 + x_2y_1)i$ طرف ايمن

$$\therefore \overline{z_1.z_2} = \bar{z}_1.\bar{z}_2$$

[6] $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$ H.W

[7] $z + \bar{z} = 2Re(z)$

Proof: Let $z = x + yi$ then $\bar{z} = x - yi$

$$z + \bar{z} = x + yi + x - yi = 2Re(z)$$

[8] $z - \bar{z} = 2i\, Im(z)$ H.W

[9] $z.\bar{z} = |z|^2$

Proof: Let $z = x + yi$ then $\bar{z} = x - yi$

$$z.\bar{z} = (x + yi).(x - yi)$$

$$= x^2 + y^2 = |z|^2$$

[10] $\frac{1}{z} = \frac{\bar{z}}{z\,\bar{\bar{z}}}$

Proof: Let $z = x + yi$ then $\bar{z} = x - yi$

$$\frac{1}{z} = \frac{1}{x + yi} . \frac{x - yi}{x - yi} = \frac{x - yi}{x^2 + y^2} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\,\bar{\bar{z}}}$$

خواص القيمة المطلقة وبرهانها :

[1] $|z_1z_2| = |z_1||z_2|$

Proof:

$$|z_1z_2|^2 = (z_1z_2).\overline{(z_1z_2)} = (z_1z_2).(\bar{z}_1.\bar{z}_2)$$

$$= |z_1|^2.|z_2|^2$$

$$\therefore |z_1z_2| = |z_1||z_2|$$

[2] $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Proof :

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{z_1}{z_2}\right)\left(\frac{\overline{z_1}}{\overline{z_2}}\right)$$

$$= \frac{z_1\overline{z_1}}{z_2\overline{z_2}} = \frac{|z_1|^2}{|z_2|^2}$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Triangle inequality

[1] $|z_1 + z_2| \leq |z_1| + |z_2|$

[2] $|z_1 - z_2| \leq |z_1| + |z_2|$

[3] $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$

[4] $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$ (H.W)

[1] proof: $|z_1 + z_2| \leq |z_1| + |z_2|$

$$|z_1 + z_2|^2 = (z_1 + z_2).\overline{(z_1 + z_2)} = (z_1 + z_2).(\overline{z_1} + \overline{z_2})$$

$$= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}$$

$$= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2$$

$$= |z_1|^2 + 2Re(z_1\overline{z_2}) + |z_2|^2$$

$$\leq |z_1|^2 + 2\,|z_1\overline{z_2}| + |z_2|^2$$

$$\leq |z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2$$

$$\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$$\leq (|z_1| + |z_2|)^2$$

[2] $|z_1 - z_2| \leq |z_1| + |z_2|$

$$|z_1 - z_2| = |z_1 + (-z_2)|$$

$$\leq |z_1| + |(-z_2)|$$

$$\leq |z_1| + |z_2|$$

[3] $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$

$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$$

$$-|z_1 + z_2| \leq |z_1| - |z_2| \leq |z_1 + z_2|$$

$$|z_1| = |z_1 + z_2 - z_2\,| \leq \ |z_1 + z_2| + |-z_2|$$

$|z_1| - |z_2| \leq |z_1 + z_2|$............(1)

$$|z_2| = |z_2 + z_1 - z_1\,| \leq \ |z_2 + z_1| + |-z_1|$$

$$|z_2| - |z_1| \leq |z_1 + z_2|$$

$$-[|z_2| - |z_1|] \leq [z_1 + z_2]$$

$|z_2| - |z_1| \geq -|z_1 + z_2|$.........(2)

Complex analysis (lecture 1)

- **Definition of complex number تعريف العدد المركب**
- **Addition and subtraction of complex numbers جمع وطرح الاعداد المركبة**
- **Multiplying complex numbers ضرب الاعداد المركبة**
- **Division of complex numbers قسمة الاعداد المركبة**
- **Proof of some Properties of complex numbers برهان بعض الخواص**
- **A negative number inside a square root عدد سالب داخل الجذر التربيعي**
- **Solving quadratic equation in ℂ حل المعادلات التربيعية في الاعداد المركبة**
- **Proof of some properties of complex number برهان بعض خواص الاعداد المركبة**
- **The scale is modules and some notes on it المقياس وبعض الملاحظات عليه**
- **The equation of a circle in complex numbers معادلة الدائرة في الاعداد المركبة**

مقدمة في الاعداد العقدية : $\mathbb{C}$

Sums and products

Definition : (1-1) : we define complex numbers as order pairs (x ,y) of real numbers x and y such that if $z = (x, y)$ then x is called real part $[Re(z) = x]$, y is called the imaginary part $[Im(z) = y]$

- If x=0 then $z = (0, y)$ is called pure imaginary number .

Example (1-1) : Let $z = (-8,4)$ then :

$$Re(z) = -8 \quad , Im(z) = 4$$

Remark (1-1):

$$\forall x \in R, \quad x = (x, 0) \in \mathbb{C}, \qquad R \subseteq \mathbb{C}$$

Definition (1-2): Let $z_1 = (x_1, y_1) and \ z_2 = (x_2, y_2)$ be two complex numbers, then :

[1] $z_1 = z_2 \ iff x_1 = x_2 \ and \ y_1 = y_2$

[2] $z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$

[3] $z_1 . z_2 = (x_1, y_1).(x_2, y_2) = \left((x_1x_2 - y_1 \ y_2), (x_1y_2 + y_1x_2 \ )\right)$

[4] $\frac{z_1}{z_2} = \frac{(x_1,y_1)}{(x_2,y_2)} . \frac{(x_2,-y_2)}{(x_2,-y_2)}$

Proposition (1-1): Let $i = (0,1)$ then $i = \sqrt{-1}$

[1] $i^2 = -1$

Proof : $i^2 = i.i = (0,1)(0,1) = (0 - 1,0 + 0) = (-1,0) = -1$

[2] Every complex number $z = (x, y)$ can be written $z = x + yi$

Proof:

$$\because z = x + yi$$
$$x \in R \Longrightarrow (x, 0) \in \mathbb{C}$$
$$y \in R \Longrightarrow (y, 0)$$
$$iy = (0,1)(y, 0) = (0 - 0,0 + y) = (0, y)$$
$$z = (x, 0) + (0, y) = (x, y)$$

Proposition (1-2): Let $z , z_1, z_2 \ and \ z_3$ be complex numbers then :

[1] $z_1 + z_2 = z_2 + z_1$

Proof:

Let $z_1 = x_1 + y_1 i \qquad and \ z_2 = x_2 + y_2 i$

$$z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i)$$
$$= (x_1 + x_2), (y_1 + y_2)i = \ (x_2 + x_1), (y_2 + y_1)i$$
$$(x_2 + y_2 i) + (x_1 + y_1 i) = \ z_2 + z_1$$

[2] $z_1 . z_2 = z_2 . z_1$

[3] $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

[4] $z(z_1 + z_2) = zz_1 + zz_2$

[5] $z + 0 = z$, المحايد الجمعي $and\ z.1 = z$ المحايد الضربي

[6] $z = x + yi$ then the additive inverse

$$-z = -(x + yi) \ \ni z + (-z) = 0$$

[7] $z = x + yi$ then the multiplicative inverse

$$z^{-1} = \frac{1}{z} = \frac{1}{x + yi} \ \ni z.z^{-1} = 1$$

<u>Example (1-2) :</u> Evaluated of the following

[1] if $z_1 = (2,1) and\ z_2 = (3,2)\ \ then\ z_1 + z_2 = ?$

$$z_1 + z_2 = (5,3)$$

[2] if $z_1 = -3 + 2i\ and\ z_2 = 4 - 5i\ \ then\ z_1 - z_2 = ?$

[3] if $z_1 = (1,2) and\ z_2 = (1,3)\ \ then\ z_1 . z_2 = ?$

$$z_1 . z_2 = (1,2).(1,3) = (1 - 6, 3 + 2) = (-5,5)$$

[4] if $z_1 = -1 - 2i\ and\ z_2 = -1 + i\ \ then\ z_1 . z_2 = ?$ H.W

[5] if $z_1 = 3 + 2i\ and\ z_2 = -1 - 2i\ \ then\ \ \frac{z_1}{z_2} = ?$

$$\frac{z_1}{z_2} = \frac{3 + 2i}{-1 - 2i} . \frac{-1 + 2i}{-1 + 2i}$$

$$\frac{-3 + 6i - 2i - 4}{1 + 4} = \frac{-7 + 4i}{5}$$

[6] if $z_1 = 4+3i \ and \ z_2 = 2-2i \ \ then \ \frac{z_1}{z_2} = ?$ H.W

Remark (1-2) : $\sqrt{-a} = \sqrt{a}i$

Example (1-3) : Evaluated of the following:

[1] $\sqrt{-16} = \sqrt{16}\,\sqrt{-1} = 4i$

[2] $\sqrt{-100} = 10i$

[3] $\sqrt{-13} = \sqrt{13}\,\sqrt{-1}\,\sqrt{13}i$

[4] $\sqrt{-3} = ?$

Example (1-4) : solve

[1] $z^2 + 2 = 0$

Solution : $z^2 = -2 \quad \Rightarrow z = \mp\sqrt{-2} = \mp\sqrt{2}\,.\sqrt{-1} \mp \sqrt{2}i$

$$S = \{\sqrt{2}i\,, -\sqrt{2}i\}$$

[2] $z^2 + 3iz - 2 = 0$ H.W

[3] $z^2 + 2z + 3 = 0$

$$z^2 + 2z + 1 + 2 = 0$$

$$z^2 + 2z + 1 = -2$$

$$(z+1)^2 = -2$$

$$z + 1 = \mp\sqrt{2}i$$

$$z = \begin{cases} -1 + \sqrt{2}i \\ -1 - \sqrt{2}i \end{cases}$$

$$S = \{1 \mp \sqrt{2}i\}$$

Example (1-5) : Show that $Im(z) = Re(z)$

Solution : Let $z = x + yi$

$$Re(z) = x \dots\dots\dots\dots(1) \; and \; Im(z) = y \dots\dots\dots(2)$$

$$iz = i(x + yi) = ix - y = -y + ix$$

$$\therefore Im(z) = Re(z)$$

Moduli and conjugates

Definition : (1-2) : modulus of a complex number

$$z = x + yi \; is \; |z| = \sqrt{x^2 + y^2}$$

Example (1-6) : Find $|z|$ if

[1] $z = 1 + 4i \quad \Rightarrow \quad |z| = \sqrt{(1)^2 + (4)^2} = \sqrt{1 + 16} = \sqrt{17}$

[2] $z = 4 - 3i$ H.W

Remarks (1-3) :

[1] For any complex number z , $|z| \in R \; and \; |z| \geq 0$

Example (1-7) : $|3i| = |0 + 3i| = \sqrt{0 + 9} = 3$

$$|1 - 2i| = ?$$

$$|i| = |0 + i| = 1$$

[2] $|z_1| < |z_2|$ means that the point z_1 is closer to the origin that point z_2
if $z_1 < z_2$ is meaningless unless both $z_1 and \; z_2$ are real >

[3] $z_1 - z_2 = (x_1 + y_1 i) - (x_2 + y_2 i)$

Then $|z_1 - z_2| =?$

Solution :

$$z_1 - z_2 = (x_1 + y_1 i) - (x_2 + y_2 i)$$

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

[4] The equation of the circle with center z_0 and radius r is $|z - z_0| = r$

Example (1-8) : [1] The equation $|z - 1 + 3i| = 2$

Solution : $|z - (1 - 3i)| = 2$

$z_0 = 1 - 3i = (1, -3)$, $r = 2$

[2] The equation $|z + 2 - i| = 4$

Solution : $|z - (2 + i)| = 4$

$z_0 = (2,1)$, $r = 4$

Prove that :

[1] $Re(z) \leq |Re(z)| \leq |z|$

[2] $Im(z) \leq |Im(z)| \leq |z|$ H.W

Proof:

$$z = Re(z) + Im(z)$$

$$|z| = \sqrt{[Re(z)]^2 + [Im(z)]^2}$$

$$|z|^2 = [Re(z)]^2 + [Im(z)]^2$$

$$[Re(z)]^2 \leq |z|^2 \Longrightarrow |Re(z)| \leq |z|$$

$$\therefore\ Re(z) \leq |Re(z)| \leq |z|$$