Geometric Series

A geometric series has the following form:

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1},$$

where a and r are fixed real numbers and $a \neq 0$. The number r is called the **ratio** of the geometric series.

Consider the nth partial sum

$$s_n = a + ar + \cdots + ar^{n-1}.$$

If $r = 1$, then $s_n = na$, and hence the geometric series diverges.

Suppose $r \neq 1$. How do we find the partial sums of the geometric series?

Partial Sums of of a Geometric Series

We have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Subtracting the second equation from the first one, we get

$$s_n - rs_n = a - ar^n \quad \text{or} \quad (1-r)s_n = a(1-r^n).$$

Consequently, for $r \neq 1$ we obtain

$$s_n = \frac{a(1-r^n)}{1-r}, \quad n = 1, 2, \ldots.$$

Convergence of Geometric Series

Recall that

$$s_n = \frac{a(1-r^n)}{1-r}, \quad n = 1, 2, \ldots.$$

If $-1 < r < 1$, then $\lim_{n\to\infty} r^n = 0$, and hence

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

If $r \leq -1$ or $r > 1$, the sequence $(r^n)_{n=1,2,\ldots}$ diverges, so the series diverges in these cases.

Conclusion. If $|r| < 1$, the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

converges and its sum is

$$\frac{a}{1-r}.$$

If $|r| \geq 1$, the series diverges.

Example. Is the series $\sum_{n=1}^{\infty}\left(\frac{\pi}{3}\right)^n$ convergent or divergent?

Solution. Since $r = \frac{\pi}{3} > 1$, the series diverges.

Example. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots.$$

Solution. We have $a = 5$ and

$$r = \frac{-\frac{10}{3}}{5} = -\frac{10}{3} \cdot \frac{1}{5} = -\frac{2}{3}.$$

Since $|r| < 1$, the geometric series converges and its sum is

$$s = \frac{a}{1-r} = \frac{5}{1-(-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 5 \cdot \frac{3}{5} = 3.$$

Note that

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

A Characterization of Geometric Series

How do we know a given series

$$\sum_{n=1}^{\infty} a_n$$

is a geometric series or not?

Suppose $a_n \neq 0$ for $n = 1, 2, \ldots$. If there exists a *constant* r such that

$$\frac{a_{n+1}}{a_n} = r \quad \text{for all} \ \ n,$$

then the series is a geometric series; otherwise, it is not a geometric series.

Example. Find the sum of the geometric series

$$\sum_{n=2}^{\infty} \frac{2^{3n}}{3^{2n}}.$$

Solution. Write $a_n = \dfrac{2^{3n}}{3^{2n}}$, $n = 2, 3, \ldots$. We have

$$a = a_2 = \frac{2^6}{3^4} = \frac{64}{81}.$$

Moreover,

$$r = \frac{a_{n+1}}{a_n} = \frac{2^{3(n+1)}}{3^{2(n+1)}} \cdot \frac{3^{2n}}{2^{3n}} = \frac{2^{3n+3}}{2^{3n}} \cdot \frac{3^{2n}}{3^{2n+2}} = \frac{8}{9}.$$

Since $|r| < 1$, the geometric series converges and its sum is

$$s = \frac{a}{1-r} = \frac{\frac{64}{81}}{1 - \frac{8}{9}} = \frac{\frac{64}{81}}{\frac{1}{9}} = \frac{64}{9}.$$

Note that

$$a^m \cdot a^n = a^{m+n} \quad \text{and} \quad \frac{a^m}{a^n} = a^{m-n},\ a \neq 0.$$

For example, $a^3 \cdot a^2 = a^5$ and $a^6/a^2 = a^4$.