Differential Equations

Lecture: The Method of Separation of Variables

Course Code: MATH-201 | **Level:** Undergraduate | **Topic:** First-Order Ordinary Differential Equations

1 Introduction and Theoretical Foundation

A first-order ordinary differential equation (ODE) relates an unknown function $y(x)$ to its derivative $\frac{dy}{dx}$. The **Method of Separation of Variables** is one of the fundamental analytical techniques for solving first-order ODEs.

> **Definition (Separable Differential Equation):**
> A first-order differential equation is called **separable** if it can be expressed in the form:
>
> $$\frac{dy}{dx} = f(x)g(y) \quad \text{or in differential form} \quad M(x)\,dx + N(y)\,dy = 0 \tag{1}$$
>
> where $f(x)$ and $M(x)$ depend *only* on x, and $g(y)$ and $N(y)$ depend *only* on y.

1.1 Key Intuition

The core idea is algebraic isolation: arrange all terms containing y and dy on one side of the equation, and all terms containing x and dx on the opposite side. Once separated, both sides are integrated independently.

2 Step-by-Step Solution Algorithm

To solve a separable first-order differential equation $\frac{dy}{dx} = f(x)g(y)$, follow these structured steps:

Step 1: Check for Singular Solutions: Identify values of y where $g(y) = 0$. If $g(y_0) = 0$, then the constant function $y(x) = y_0$ is an equilibrium solution.

Step 2: Separate the Variables: Divide by $g(y)$ (assuming $g(y) \neq 0$) and multiply by dx:

$$\frac{1}{g(y)}\,dy = f(x)\,dx$$

Step 3: Integrate Both Sides: Apply indefinite integration to both sides:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Step 4: Combine Constants of Integration: If $H(y) = \int \frac{1}{g(y)} dy$ and $F(x) = \int f(x) dx$, write:

$$H(y) = F(x) + C$$

Step 5: Solve for y (Explicit Form, if possible): Isolate y to obtain an explicit solution $y = \phi(x, C)$. Otherwise, leave as an implicit relation $H(y) - F(x) = C$.

Step 6: Apply Initial Conditions (For IVPs): Substitute given initial values $y(x_0) = y_0$ to solve for C and obtain the unique particular solution.

Note on Singular Solutions:
When separating variables by dividing by $g(y)$, setting $g(y) = 0$ yields constant solutions that might not be obtainable for any finite choice of C in the general solution.

3 Comprehensive Worked Examples

3.1 Example 1: Basic Algebraic Separation

Problem: Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{1+y^2}$.

Solution:

1. Separate variables: $(1 + y^2)\,dy = x^2\,dx$
2. Integrate both sides: $\int(1 + y^2)\,dy = \int x^2\,dx$
3. Perform integration: $y + \frac{y^3}{3} = \frac{x^3}{3} + C$
4. Simplify by multiplying by 3: $3y + y^3 = x^3 + C' \quad (\text{where } C' = 3C)$

This provides the complete implicit general solution.

3.2 Example 2: Initial Value Problem (IVP)

Problem: Solve the initial value problem $\frac{dy}{dx} = -2xy$, with $y(0) = 5$.

Solution:

1. Check equilibrium solution: $y = 0$ gives $\frac{dy}{dx} = 0$, which satisfies the ODE.
2. Separate variables for $y \neq 0$: $\frac{1}{y}\,dy = -2x\,dx$
3. Integrate both sides: $\int \frac{1}{y}\,dy = \int -2x\,dx \implies \ln|y| = -x^2 + C_1$
4. Exponentiate both sides: $|y| = e^{-x^2+C_1} \implies y(x) = Ae^{-x^2}$ where $A = \pm e^{C_1} \in \mathbb{R}$.
5. Apply initial condition $y(0) = 5$: $5 = Ae^0 \implies A = 5$.
6. Particular Solution: $y(x) = 5e^{-x^2}$.

3.3 Example 3: Real-World Application — Newton's Law of Cooling

Application Context: Newton's Law of Cooling states: $\frac{dT}{dt} = -k(T - T_m)$, where $T(t)$ is temperature, T_m is ambient temperature, and $k > 0$. A hot liquid at 90°C is placed in a 20°C room. After 10 minutes, it cools to 60°C. Find $T(t)$.

Solution:

1. Substitute ambient temperature $T_m = 20$: $\frac{dT}{dt} = -k(T - 20)$
2. Separate variables and integrate: $\int \frac{dT}{T-20} = \int -k\,dt \implies \ln|T - 20| = -kt + C$
3. Express explicitly: $T(t) = 20 + Ae^{-kt}$
4. Apply $T(0) = 90$: $90 = 20 + A \implies A = 70 \implies T(t) = 20 + 70e^{-kt}$
5. Apply $T(10) = 60$: $60 = 20 + 70e^{-10k} \implies e^{-10k} = \frac{4}{7} \implies k = -\frac{1}{10}\ln\left(\frac{4}{7}\right) \approx 0.0560 \text{ min}^{-1}$
6. Final Temperature Model: $T(t) = 20 + 70e^{-0.0560t} \quad (°\text{C})$

4 Extension: Separation of Variables in Partial Differential Equations (PDEs)

The principle of variable separation extends to linear PDEs such as the 1D Heat Equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{2}$$

By assuming a product ansatz $u(x,t) = X(x)T(t)$, partial derivatives yield:

$$X(x)T'(t) = \alpha X''(x)T(t) \implies \frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

where λ is the separation constant. This splits the PDE into two coupled ODEs:

$$1.\ T'(t) + \alpha\lambda T(t) = 0 \qquad \text{and} \qquad 2.\ X''(x) + \lambda X(x) = 0$$

5 Summary Table

primary!15 **Feature**	**Separable Form**	**Non-Separable Form**
General Form	$\frac{dy}{dx} = f(x)g(y)$	$\frac{dy}{dx} = f(x,y)$ (e.g., $x+y$)
Differential Form	$M(x)dx + N(y)dy = 0$	$P(x,y)dx + Q(x,y)dy = 0$
Primary Technique	Direct integration after division	Integrating factors / Substitution
Equilibrium Solutions	Roots of $g(y) = 0$	Fixed points / Nullclines

Table 1: Key Characteristics of Separable vs Non-Separable Differential Equations

6 Practice Problems

Solve the following problems to test your understanding:

1. **General Solution:** Find the general solution of $\frac{dy}{dx} = \frac{xe^x}{y\cos y}$.
2. **Initial Value Problem:** Solve $\frac{dy}{dx} = \frac{1+x}{y^2}$ subject to $y(0) = 1$.
3. **Population Model:** Solve $\frac{dP}{dt} = kP$ with $P(0) = P_0$. If $P(0) = 1000$ and $P(2) = 3000$, calculate the time t when $P(t) = 4000$.
4. **Singular Solution Analysis:** Find all solutions of $\frac{dy}{dx} = y^{2/3}$. Discuss why $y(x) = 0$ is a singular solution.

End of Lecture Notes — Mathematics II (Differential Equations)